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Application of Mathematical models of Moving Particles for Transport Systems

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Abstract

In the paper mathematical model of moving particles on a circle which can be used for transport systems is considered. In the capacity of efficiency index is taken an average waiting time of particle at the fixed point of a circle. Such a model can be applied for analyzing transport systems for finding an optimal number of transport units, optimizing an operation of a system. Numerical examples demonstrated theoretical results are given.

Keywords: motion on a circle; absolute stable regime; saturated regime; optimal number of moving particles; efficiency index; diagram of a motion

Introduction

Mathematical models of moving particles are widely used for various applications, in transport systems, for investigation of communication systems, in biology and others. Construction and investigation of mathematical models of moving particles allows us to predict and avoid the traffic jams [6]. In [3, 7] mathematical models of moving particles on a straight line constructed and investigated. It was obtained so called Belyaev's effect, when separately considered particle makes a random binomial walk. This fact has been used for prediction and avoiding traffic jams in some systems. In [4] the Belyaev's effect has been generalized for motion on a circle. All models considered in [3, 4, 7] related to discrete motion of particles. The continuous motion of particles on a circle allows more adequate to describe the behavior of transport systems but unfortunately, they have rather complicated structure, and its investigation desire the creation new approaches and methods for their investigation. In this direction using simulation and data analysis of results of simulation allows us to take effective decisions and recommendations for applications [2]. One of a main problems is to find an optimal number of particles which optimizes chosen efficiency index.

Construction of the mathematical model

Consider moving S particles on a circle of length 1 in the clockwise direction. At each point of a circle, there can be only one particle. The particles are numbering in the direction of a movement. Each particle can move with speed V_1 or V_2, where $V_1<V_2$. Introduce the following notations:

$x_{i,t}$ - is the coordinate of the *i-th* particle at the instant *t*.

$V_{i,t}$ - is the speed of *i-th* particle ($i = 1,2,\dots,S$) at the instant *t*;

$r_{i,t}$ i-is the distance (in the direction of a movement) between *i-th* and (i+1) particles, which is defined as

$$\rho_{i,t}=\begin{pmatrix} x_{i+1,t}-x_{i,t}, & if\ x_{i+1,t}>x_{i,t} \\ 1-\left(x_{i,t}-x_{i+1,t}\right), & if\ x_{i+1,t}<x_{i,t} \end{pmatrix}\ i=1,2,..,s-1;$$

$$\rho_{1,t}=\begin{pmatrix} x_{1,t}-x_{s,t}, & if\ x_{1,t}>x_{s,t} \\ 1-\left(x_{s,t}-x_{1,t}\right), & if\ x_{1,t}<x_{s,t} \end{pmatrix}$$

Introduce the values Q_1, Q_2 and assume $Q_1 < Q_2 < 1$, where Q_1 and Q_2 define the speed of particles. It is assumed that $\frac{Q_2}{V_2}<\frac{Q_1}{V_1}$. This assumption means that for transport systems it is preferable to keep long distance between units and have high speed.

If $\rho_{i,t}\geq Q_2$ then $V_{i,t}=V_2$ and if $\rho_{i,t}=Q_1$, then $V_{i,t}=V_1$ If $\rho_{i,t}=Q_1$ and the distance $\rho_{i,t}$ is increasing then at the instant t^*, when $\rho_{i,t^*}=Q_2$ the *i-th* particle changes the speed from V_1 to V_2. If $\rho_{i,t}\geq Q_2$ and the distance $\rho_{i,t}$ is decreasing then at the instant t^{**}, when $\rho_{i,t^{**}}=Q_1$ the *i-th* particle changes the speed from V_2 to V_1. Such picture is observed in transport systems when long distance between transport units allows to have high speed and vice versa, when short distance between moving units is forced to keep low speed.

Thus, the particles change their speed if the distance between them becomes equal Q_1 or Q_2, i.e. a speed is changing only on the border of the interval $[Q_1, Q_2]$. Hence, if a distance between particles takes value from the interval (Q_1, Q_2) then a particle does not change its speed and keeps either the speed V_1 or V_2.

In the capacity of the efficiency index is taken an average *waiting time* of *particle* at the fixed point of a circle, which is denoted *wtp(.)*. Other words, we take randomly (uniformly) some point on a circle and calculate an average waiting time of particle, which passes this point. In transportation systems (vertical transportation, public transport, underground trains) it can be explained as a customer average waiting time before service, in communication systems as average waiting time for getting information.

Remark 1

In this model instead of a circle any closed trajectory can be considered, for instance broken line, but for simplicity we consider motion of particles on a circle of length 1. In fact, movement of buses, underground trains and other public transportation can be described by the motion of particles on a closed curve or broken line, because all of them have closed paths.

Definition 1

If for some t the distances between (*s-1*) particles equal Q_1, then this state is called the *zero state*.

We consider stationary regime of motion, where $t \to \propto$.

Definition 2

If for some *t*, all particles have the speed V_1 ($V_{i,t} = V_1$, $i=1,2,..s$;) then this state of motion is called *Absolutely Stable State1* (*ass1*).

If system starts from the zero state and there exists instant t^*, when all particles move with the speed V_1 ($V_{i,t}{}^*=V_1$, $i=1,2,\dots,s$), then for any $t > t^*$, is held $V_{i,t}=V_1$, $i=1,2,..,s$ and system operates in the regime of motion (*ass1*).

Remark 2

For given S, Q_1 and Q_2 the regime of motion (*ass1*) does not always exist. For instance, if S satisfies the equation $(S\text{-}1)Q_1+Q_2 = 1$ and $2Q_1>Q_2$, then for such S the regime of motion (*ass1*) does not exist, because there exist some particles and the distance between them is greater than Q_2, hence according to construction of the model, one particle must have the speed V_2. Even changing (increasing) the number of particles S we can't get the regime of motion (*ass1*), because there is no possibility of adding any new particle.

Theorem 1

For given Q_1 and Q_2, starting from the *zero state*, for existing of the regime of motion (*ass1*) with S - unique number of particles it is necessary and sufficient.

$$1 - min(Q_2\text{-}Q_1,Q_1)< sQ_1 \le 1 \text{ or equivalent form}$$

$$1 - min[(Q_2,2Q1)\text{-}Q_1]< sQ_1 \le 1 \ (1)$$

Proof

Necessity. If system starts from the zero state and reaches the regime of motion (*ass1*), then distance between (s-1) particles exactly equal Q_1 and distance between other remaining two particles must be less than $2Q_1$, but greater than Q_1, because otherwise (*ass1*) is not unique (for instance we can have also (*ass1*) with (s+1) number of particles). It follows, that.

$$sQ_1 \le 1 \text{ and } (s\text{-}1)Q_1+2Q_1>1 \ (2)$$

At the same time this distance must be strongly less than Q_2, because otherwise it will be mixed regime (if there exist at least two particles, that distance between them equals or greater than Q_2, then it can be the regime of motion (*mss*)), i.e. it should be held.

$$(s\text{-}1)Q_1+Q_2>1. \ (3)$$

From (2) and (3) follows (1).

Sufficiency follows directly from (1). As the system starts from the *zero state* and (1) is held, then the distances between (s-1) particles equal exactly Q_1 and hence $V_{i,t}=V_1$, $i=1,2,..,s\text{-}1$; but from (1) follows that for last one is held $Q_1 < r < min(Q_2, 2Q_1)$. Hence the last particle also has speed V_1.

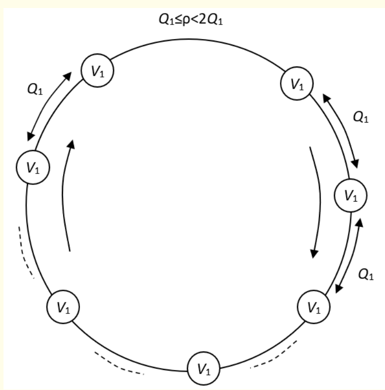

Figure 1: Regime *ass1*.

Thus, from (2) and (3) for existing of regime of motion (*ass1*) for given Q_1 and Q_2 the number of particles is defined by the following equation.

$$s = \{max \text{ integer } k: (1-Q_2+Q_1, 1-Q_1 < kQ_1 \leq 1\} \quad (4)$$

If $Q_2 > 2Q_1$, then $1 - (Q_2 - Q_1) < sQ_1 \leq 1$

and if $Q_2 \leq 2Q_1$, then $(1 - Q_1) < sQ_1 \leq 1$.

It is easy to calculate that for (*ass1*) regime of motion is true.

$$wtp_{ass1} = [(s-1)Q_1^2+z_1^2]/2V_1 \quad (5)$$

where $Q_1 \leq z_1 < 2Q_1$

Thus, $sQ_1^2/2V_1 \leq wtp_{ass1} < (s+3)Q_1^2/2V_1$ (5*)

If $z \approx Q_1$, then we can put $sQ_1 \approx 1$ and hence, we have

$$wtp(ass1) = Q_1/2V_1 \quad (6)$$

Remark 3

If condition (1) is not held, then we can have several (*ass1*) with different number of particles.

Example 1

$Q_1=1/30$; $Q_2=1/10$; $s_1=30$; hence, (*ass1*) with 30 particles exists.

Consider case S_2 = 29; then there exists t for which is held

$r_{1,t} = 1/30, V_{1,t} = V_1; \rho_{2,t} = 1/30, V_{2,t} = V_1, \ldots, \rho_{28,t} = 1/30, V_{28,t} = V_1; Q_1 = 1/30 < r_{29,t} = 2/30 < Q_2 = 1/10, V_{29,t} = V_1;$

Definition 2

The regime is motion, where all particles have the speed V_2 ($V_{i,t} = V_2$, $i=1,2,..s$) is called ***Absolutely Stable State2***, (***ass2***).

If system starts from the zero state and there exist some instant t*, when all particles have the speed V_2 i.e. ($V_{i,t}^* = V_2$, $i=1,2,\ldots,s$) then for any $t > t^*$, $V_{i,t}=V_2$, $i=1,2,\ldots,s$; and the system operates in the regime of motion (***ass2***).

As each particle in the system can change its speed only at the instant, when the distance between moving particles becomes Q_1 or Q_2, then it follows from construction of the model that if system starts motion from the zero state then for (*ass2*) is true:

a. All distances between particles, except only one, should be equal Q_2 and only for one of them (for instance, between *s-th* and first particle) must be $Q_1 < r_{s,t} < 2Q_1$, (see, Fig. 2)
b. Unlike of the regime of motion (*ass1*), for given Q_1 and Q_2 under the condition (*a*), there exist various regimes of motion (*ass2*) with different number of particles, i.e. for given Q_1 and Q_2 for regime of motion (*ass2*) the number of particles s is not unique.

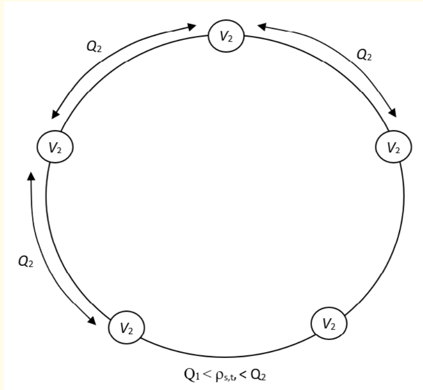

Figure 2: Regime *ass2*.

Remark 4

For given Q_1 and Q_2 the regime of motion (*ass2*) is not unique. There exists various regimes of motions (*ass2*) with different number of particles.

Example 2

Suppose $Q_1 = 1/20$; $Q_2 = 1/5$; s = 5;

$r_{1,0} = Q_1, V_{1,0} = V_1; r_{2,0} = Q_1, V_{2,0} = V_1; \dots, r_{4,0} = Q_2, V_{4,0} = V_1;$

$r_{5,0} = 1 - 4/20 = 4/5; V_{5,0} = V_2;$

Then after the time $t^* = 4(Q_2 - Q_1)/(V_2 - V_1)$ we have

$r_{1,t^*} = 1/5, V_{1,t^*} = V_2; r_{2,t^*} = 1/5, V_{1,t^*} = V_2; r_{3,t^*} = 1/5, V_{3,t^*} = V_2$

$r_{4,t^*} = 1/5, V_{4,t^*} = V_2; r_{5,t^*} = 1/5, V_{5,t^*} = V_2;$

i.e. we have regime of motion (*ass2*) with $s = 5$.

If $Q_1 = 1/20$; $Q_2 = 1/5$; $s = 3$;

and

$r_{1,0} = Q_1, V_{1,0} = V_1; r_{2,0} = Q_1, V_{2,0} = V_1; r_{3,0} = 9/10 > Q_2, V_{3,0} = V_2;$

After time $t^{**} = 2(Q_2 - Q_1)/(V_2 - V_1)$ we have

$r_{1,t^{**}} = Q_2, V_{1,t^{**}} = V_2; r_{2,t^{**}} = Q_2, V_{2,t^{**}} = V_2;$

$r_{3,t^{**}} = (1 - 1/10) = 9/10 > Q_2, V_{3,t^{**}} = V_2;$

Thus, here is the regime of motion (*ass2*) with $s = 3$.

Remark 5

If for given Q_1 and Q_2 there exists some S, that system operates in the regime of motion (*ass2*), then for any $S^* < S$ the system will operate also in the regime of motion (*ass2*).

If we are interested in the regime of motion (*ass2*) with possibly maximum number of particles, then for given Q_1 and Q_2 the number of particles (s) is unique and $S = \{max \text{ integer } k\text{: } kQ_2 \leq 1\}$.

Another example of the regimes of motion (*ass2*) is the following

$r_{1,t} = k_1Q_2, V_{1,t}=V_2; r_{2,t} = k_2Q_2, V_{2,t}=V_2; \ldots, r_{s\text{-}1,t} =k_{s\text{-}1}Q_2, V_{s\text{-}1,t} = V_2;$

$r_{s,t} = k_sQ_2, V_{s,t} = V_2; k_i>1, i=1,2,..,s.$

But such regimes of motion cannot be reached from the zero state and we will consider only the regimes of motion, which can be reached from the zero state and moreover it is assumed that both states (*ass1*) and (*ass2*) are the regimes with maximal possibly number of particles, i.e. the number of particles for (*ass2*) must satisfy to the following expressions.

$$0<sQ_2 \leq 1, (s+1)Q_2 >1 \quad (7)$$

$$sQ_2+Q_1 >1 \quad (8)$$

From (2.6.7) and (2.6.8) it follows

$$S = \{max \text{ integer } k\text{: } (1\text{-}Q_2, 1+Q_2 \text{-}Q_1)< kQ_2 \leq 1\} \quad (9)$$

If Q_2 is given, then the condition (9) defines unique s for (*ass2*). Other words, we consider (*ass2*) with maximum possibly number of particles, which satisfies to the condition (9).

Similarly, as the formula (5) for (*ass2*) we have

$$wtp_{ass2} = [(s\text{-}1)Q_2^{\,2} + z_1^{\,2}]/2V_2 \quad (10)$$

$$\text{where } Q_1< z \leq Q_2 \quad (10^*)$$

Thus,

$wtp_{ass2} = [(s\text{-}1)Q_2^{\,2} + z_2^{\,2}]/2V_2 < [Q_2 - Q_2^{\,2}+4Q_1^{\,2}]/2V_2.$

As $Q_1 < (1\text{-}(s\text{-}1)Q_2) \leq Q_2$ the following formula is true

$$(s\text{-}1)Q_2/2V_2+Q_1/2V_2 < wtp_{ass2} \leq Q_2/2V_2 \quad (11)$$

Remark 6

It is necessary to note that for regimes of motion (ass1) and (ass2) the number of the particles (s) are different, and moreover.

$S_{(ass1)} > S_{(ass2)}.$

The statement of Remark 3 follows from (4) and (9).

We assume $Q_2/V_2 < Q_1/V_1$. It means that the regime of motion (*ass2*) is preferable than the regime of motion (*ass1*) because from (6) and (11) we have the following inequality.

$$wtp_{(ass2)} < wtp_{(ass1)} \quad (12)$$

i.e. the regime of motion (*ass2*) is preferable (relatively to chosen efficiency index) than (*ass1*).

Remark 7

From (12) it follows that in spite of the fact, that in the regime of motion (*ass1*) the number of the particles always is greater than in (*ass2*), nonetheless, the regime (*ass2*) is preferable, than the regime (*ass1*), relatively to chosen efficiency index $wst_{(.)}$, because as it was shown (12) $wtp_{(ass2)} < wtp_{(ass1)}$.

Definition 3

The regime is motion, where some particles have the speed V_1 ($V_{i,t} = V_2$, $i=1,2,..k_1$) and others V_2 ($V_{j,t} = V_2$, $j=k_1,k_1+1,..,s$) is called ***Mixed Stable State***, ***(mss)*** and is denoted $\boldsymbol{M(k_1,k_2)}$.

If the system is operating in the regime of motion (*mss*), then some time it operates in $M(k_1,k_2)$, sometime $M(k_1-1,k_2+1)$ and some time $M(k_1+1,k_2-1)$, but $k_1+k_2=s$. Later, it will be given the diagram, allowing to find k_1 and k_2 for given s, Q_1 and Q_2 for system, which starts from the zero state.

Now, for regime of motion (*mss*) we have to define two remaining distances between groups of particles in the mixed regime of motion. It is clear, that these distances will take values between Q_1 and Q_2 i.e. $Q_1 < r < Q_2$.

Denote two, above mentioned, remaining intervals z_1 and z_2. As in the Mixed Stable State (*mss*), the intervals between (k_1-1) particles equal Q_1, between other (k_2-1) particles equal Q_2 and for two remaining intervals z_1 and z_2 is held $Q_1 < z_1,z_2 < Q_2$, then for chosen efficiency index, similarly to the formulas (5) and (9) we have

$$wst_{mss} = [(k_1-1)Q_1^2+(Q_1+z_1)^2]+(k_2-1)Q_2^2+(Q_2 - z_2)^2]/2 \quad (13)$$

If z_1 and z_2 take small values, then neglecting by them and taking into consideration that we can put $k_1Q_1+k_2Q_2\approx 1$ we have

$$wst_{mss} = k_1Q_1^2/2V_1+k_2Q_2^2/2V_2 \quad (14)$$

Corollary 1. If $k_2=0$, then from (14) follows (6) and if $k_1=0$, then from (14) follows (11).

Proof. If $k_2= 0$, then taking into consideration that $k_1+k_2=s$ and $sQ_1\approx 1$, from (13) we have.

$wst_{mss} = sQ_1^2/2V_1=(sQ_1)2Q_1/2V_1=Q_1/2V_1$.

i.e. regime of motion $M(0,s)$ coincides with (*ass2*).

Note that above the sign (≈)means approximately equals.

Similarly, if $k_1 =0$, then taking into consideration that $k_1+k_2=s$ and $sQ_2=1$ from (2.6.13) we have

$$wst_{mss} = sQ_2^2/2V_2=(sQ_2)Q_2/2V_2=Q_2/2V_2$$

i.e. the regime of motion $M(s,0)$ coincides with (*ass1*).

Remark 8

Assume that for some fixed s, Q_1, Q_2 the system is operating in the regime of motion $M(k_1,k_2)$. Consider another regime of motion $M(k_1^*,k_2^*)$, with the same values of Q_1, Q_2, where $k_2^*>k_2$. Then $wst_{M(k1^*,k2^*)} < wst_{M(k1,k2)}$.

Proof

Denote s the number of particles in the regime of motion $M(k_1,k_2)$ and s^* the number of particles in the regime $M(k_1^*,k_2^*)$. It is clear, that $s^* < s$ and $k_1^*<k_1$, because $k_1^* = max\{s: sQ_1 \leq (1-k_2^*Q_2)\}$, $k_1 = max\{s: sQ_1 \leq (1-k^2Q_2)\}$ and $(1-k_2^*Q_2) < (1-k_2Q_2)$.

Assume $k_1^*={}_{k1}-m_1$ and $k_2^*=k_2+m_2$. Then

$$m_2Q_2 \leq m_1Q_1 \quad (15)$$

and we have

$wtp_{M(k1,k2)}= k_1Q_1^2/2V_1+k2Q_2^2/2V_2$;

$wtp_{M(k1^*,k2^*)} =k_1^*Q_1^2/2V_1+k_2^*Q_2^2/2V_2 =k_1Q_1^2/2V_1-m_1Q_1^2/2V_1+$

$+ k_2 Q_2^2/2V_2+m_2 Q^{22}/2V_2 < [k_1Q_1^2/2V_1 + k_2 Q_2^2/2V_2]- m_1Q_1^2/2V_1$

$+m_2 Q_2Q_1 /2V_1= wst_{M(k1,k2)} +Q_1/2V_1 (m_2Q_2- m_1Q_1)$.

From (2.6.15) follows $wtp_{M(k1^*,k2^*)} < wtp_{M(k1,k2)}$

If $k_1^* = 0$, then instead of $M(k_1^*,k_2^*)$ we have the regime of motion (*ass2*) and it follows from the statement of the remark 2.6.6 that $wtp_{ass2} \leq wtp_{M(k1,k2)}$.

If $k_2^* = 0$, then instead of $M(k_1^*,k_2^*)$ we have the regime of motion (*ass1*) and it follows from the statement of the remark 2.6.6 $wtp_{ass1} \leq wtp_{M(k1,k2)}$.

Remark 9

It is necessary to note that formula (14) is true if the following expressions are held

$(k_1+1)\ Q_1+(k_2+1)Q_2 < 1$ and $(s+1)Q_2 > 1$.

Below, in Fig. 3 there is given the diagram, which allows for given s, Q_1 and Q_2 to find the regime of motion, where the system operates. Note that in the diagram $k_1 = s - k_2$. In the diagram the axis x corresponds to the values of Q_1 and the axis y corresponds to the values of Q_2. As $Q_1 < Q_2$, hence we are interested only in the part of the diagram, which is located upper of the graph $y = x$.

If s is given and the point (Q_1, Q_2) is located in the triangle between the lines $y = x$, $y = (1-x)/(s-1)$ and axis y, then system operates in the regime of motion (*ass1*).

If the number of particles s is given and the point (Q_1,Q_2) is located between the lines $y = x$, $y = 1- x$ and $y = -sx +1$ then system operates in the regime of motion (*ass2*).

If the number of particles s is given and the point (Q_1,Q_2) is located between the lines $y = - (s - k_2)x/k_2+1/k_2$, $y = -(s - k_2+1)x/(k_2-1) + 1/(k_2-1)$ and axis y then system operates in the regime of motion (*mss*), $M(k_1,k_2)$, where k_1 particles are moving with the speed V_1 and k_2 particles are moving with the speed V_2.

Thus, for given s, Q_1 and Q_2 the diagram given in Fig.2.6.3 allows to define regime of motion, where system is operating, starting from the *zero state*. It is clear that any Absolutely Stable State (*ass1*, *ass2* and *mss*) depends on a number of moving particles and the values of Q_1 and Q_2.

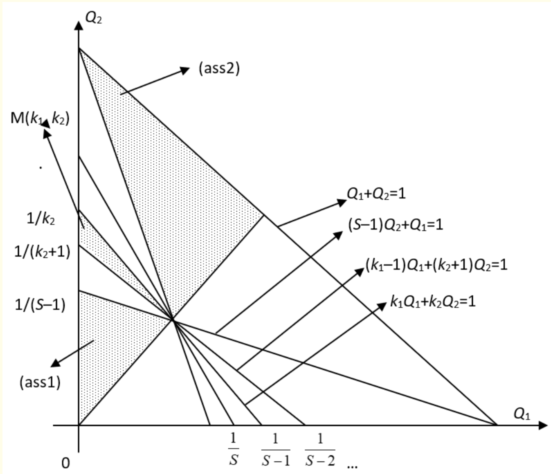

Figure 3: Diagram allowing to define the Stable States of moving particles.

Definition 4

The regime of motion with a maximum possible numbers of the particles, which move with the maximum speed V2, where a distance between them are minimal, i.e. $r_{i,t} = Q_1+e_i$, $e_i > 0$ takes a small value is called ***Saturated Stable State*** (***sss***), see, Fig. 4.

As it is assumed that the number of particles s is a maximal possible number, then it follows, that $Q_1 < e_1+e_2 +...+e_s < Q_2$, because otherwise we can have (*s+1*) particles in the system, but it contradicts to the assumption that system is operating in the (*sss*), with s particles. see, Fig.4 below.

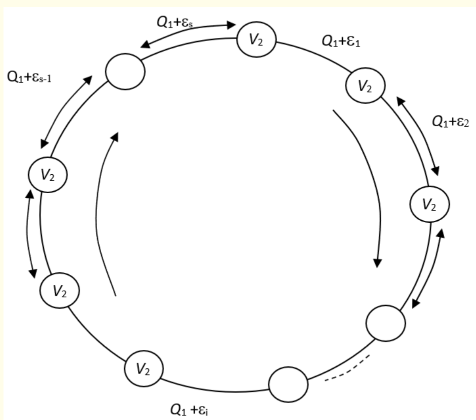

Figure 4: Saturated stable state (*sss*).

For ***Saturated Stable*** State should be held the following expressions

$sQ_1 < 1$ and $(s-1)Q_1+Q_2>1$

because from construction of this regime of motion it follows that the distances between particles is greater than Q_1 and less than Q_2. It means that in this regime of motion in the system the number of particles as much as possible, but the distances between particles possibly minimal. It is assumed that e_i takes small values, then neglecting by $e_i \approx 0$ we have

$wtp_{sss}=(Q_1+e_1)^2/2V_2+(Q_1+e_2)^2/2V_2+...+(Q_1+e_s)2/2V_2\ sQ_1^{\ 2}/2V_2\approx$

$\approx Q_1/2V_2$

and taking into consideration that for (*sss*) regime of motion we can put $sQ_1\approx 1$ and as $Q_1 <Q_2$, then we have

$Q_1/V_2 < Q_2/V_2 = wtp_{ass2}$.

Thus, $wtp_{sss} < wtp_{ass2}$.

We assume that for $t = 0$ (the zero state)

$r_{1,0}=r_{2,0} =...=r_{s,0}=Q_1$ and $r_{s,0}= 1- (s-1)Q_1$

Starting from the zero state the system can reach (*ass1*), (*ass2*) and (*mss*) regimes of motion, but not (*sss*). For (*sss*) regime of motion it is necessary to put distance between particles a little bit longer than Q1 and force them to have the speed V_2. It is not contradictory to construction of the model and hence, such regimes of motion can exist. Moreover, in practice, for instance, in public transportation, underground trains and others we face such unwished regimes of motion.

If the system starts from above mentioned the zero state, then for reaching the regime (*ass1*) it is necessary to have number of particles, which is defined by (4). This number is unique for (*ass1*). For establishing the (*ass2*) regime of motion from the zero state the number of particles is not unique. For instance, if s is a number of particles, when the system starting from zero state reaches (*ass2*) regime of motion, then the regime (*ass2*) also can be reached for any $s^*<s$. If for some s and t is true

$r_{1,t} = Q_2,\ V_{1,t} = V_2;\ r_{2,t} = Q_2,\ V_{2,t} = V_2;...,\ r_{s-1,t} = Q_2,\ V_{s-1,t} = V_2;$

$r_{s,t} =kQ_2,\ V_{s,t} = V_2,\ k\geq 1.$

then as $(1-(s-1)Q_2) = kQ_2 \geq Q_2$ hence, for any $s^* < s$ there exists some $t^*<t$ such that

$r_{1,t^*} = Q_2,\ V_{1,t^*} = V_2;\ r_{2,t^*} = Q_2,\ V_{2,t^*} = V_2;...,\ r_{s^*-1,t} = Q_2,$

$V_{s^*-1,t} = V_2;\ r_{s^*,t} = Q^*,\ V_{s,t^*} = V_2$, where $Q^* = (1-(s^*-1)Q_2)>$

$>(1-(s-1)Q_2) \geq Q_2$.

Below we will consider only the regime of motion (*ass2*) with maximum number of particles, when distance between ($s-1$) particles equal Q_2 and only between two particles the distance takes values, which are greater than Q_1.

Theorem 1

If $Q_2/V_2 < Q_1/V_1$ then the regime of motion (*sss*) is an optimal regime i.e. the following relation is true

$wtp_{sss} < wtp_{ass2} < wtp_{M(k,k)} < wtp_{ass1}$

Proof

Compare chosen efficiency index (an average waiting time of a particle at some point of a circle) for various regimes of motion. Denote Q_i* distance between *i-th* and (*i+1*) particle in the regime of motion (*sss*). As $Q_i^*<Q_2$ then we have

$wtp_{sss} = (Q_1+e_i)/2V_2 < Q_2/2V_2 = wtp_{ass2}$

Hence, $wtp_{sss} < wtp_{ass2}$.

The relation $wtp_{ass2} \leq wtp_{M(k,k)} \leq wtp_{ass1}$ follows from Remark 6.

Consider $wtp_{sss} = Q_1/2V_2 < Q_2/2V_2 = wtp_{ass2}$

$$\text{i.e. } wst_{sss} < wst_{ass2} \quad (16)$$

From Remark 6 and (16) follows

$$wtp_{sss} < wsp_{ass2} \leq \mathrm{wtp}_{(k,k)} \leq wtp_{ass1} \quad (17)$$

It seems that in practice the best regime of a motion is the saturated established state. Unfortunately, all real systems have stochastic structure (not deterministic) and the result of Theorem 1 can't apply directly for real systems but it can be some hint for planning schedule for various transportation systems.

If Q_1 and Q_2 are given, then for deterministic systems, when system starts from initial (zero) state, we can construct the diagram, which allows to define all regimes of motion (*ass1*), (*ass2*) and (*mss*) (*except* (*sss*)) in which system is operating. Below (see, Fig.3) there is given the diagram for defining the regimes of motion of the system.

Denote that the point (Q_1,Q_2) can be located only upper of the graph $Q_1=Q_2$, because *it* is assumed that $Q_1 < Q_2 < 1$. For given values Q_1 and Q_2 using the diagram (see, Fig.3) we can define a regime of motion of the system by the following way. If the point (Q_1,Q_2) is located between the lines ($k_1Q1+k_2\ Q_2$=1) and ($(k_1-1)Q_1+k_2+1)Q_2=1$), then we can conclude that the system is operating in the mixed regime $M(k_1,k_2)$, where during the time interval.

$$(1 - (k_1Q_1+k_2\ Q_2))/(V_2-V_1) \quad (18)$$

System it is operating in the state $M(k_1,k_2)$ and during the time.

$$(Q_2 - Q_1)/(V_2-V_1) \quad (19)$$

in state $M(k_1-1,k_2+1)$.

The formulas (18) and (19) have the following explanation.

As $Q_1 < (1-(k_1Q_1+k_2Q_2)) < Q_2$ and

the distance between particles is changing with the speed (V_2-V_1), because one particle has the speed V_2 and another V_1. Hence, for changing the speed from V_2 to V_1 it needs the time

$(1-(k_1Q_1+k_2Q_2))/(V_2-V_1)$.

Similarly, for changing speed from V_1 to V_2 it takes the time

$(Q_2-Q_1)/(V_2-V_1)$.

Remark 10

The diagram given in Fig.3 describes only the regimes of motion, which can be reached starting from the zero state, i.e. the regimes of motions (*ass1*), (*ass2*) and (*mss*). As the regime of motion (*sss*) cannot be reached from the zero position, then this regime of motion is not defined by diagram 3. At the same time the regime (*sss*) satisfies all, above-mentioned conditions, and hence it is not contradictory to constructed mathematical models.

Remark 11

The regime (*ass2*) unlike the regime (*ass1*) can have different numbers of particles but we are interested in (*ass2*) with possibly maximal numbers of particles. Hence, it is assumed that for (*ass2*) the condition (9) is always held.

Remark 12

We assume also that for $k_1 \geq 1$, $k_2 \geq 1$ the following condition is held

$$wtp_{ass2} \leq wtp_{M(k1,\, k2)} \quad (20)$$

although there are exist some pairs of values Q_1, V_1 and Q_2, V_2 for which the expression is not true. Hence, we will consider only such pairs of values Q_1, V_1 and Q_2, V_2, for which the condition (7) is held.

Theorem 2.

For stationary regime of motion for constructed models we have:

If system starts from the zero state, then:

1. The regime of motion (*ass1*) can be reached if and only if

$$[1-(Q_2-Q_1)]/Q_1 < s \leq 1/Q_1 \quad (21)$$

   (equivalent form of (21) is
   $Q_1 \leq (1 - (s-1)Q_1) < Q_2$
2. The regime of motion (*ass2*) can be reached if and only if

$$(1-Q_1)/Q_2 < s < [1+(Q_2-Q_1)]/Q_2 \quad (22)$$

   (equivalent form of (22) is $Q_1 < (1 - (s-1)Q_2) < Q_2+Q_1$
3. In all other cases the system reaches the (*mss*) regime of motion $M\,(k_1,k_2)$, where k_1 and k_2 will be defined by given values Q_1 and Q_2 according to the diagram, (see Fig. 3) by the following conditions
   $k1Q1+k2Q2 \leq 1,\ k1Q1+(k2+1)Q2 \geq 1$

$$1-Q_2 \leq k_1Q_1 + k_2Q_2 \leq 1 \quad (23)$$

   and for given s, Q_1 and Q_2 we have

$$k_2 = max\,\{k: (k-1)\, Q_2+Q_1 < 1 \text{ and } k_1=s-k_2 \quad (24)$$

and

$$S_{ass2} < S_{mss} < S_{ass1} \quad (25)$$

Proof.

Sufficiency. It is clear that, if $r_{s,0}=Q_1$, then system is operating in the regime of motion (ass1). From the condition (21) it follows

$Q_1 \leq r_{s,0} = Q^* = 1-(s-1)Q_1 < Q_2$

If $V_{1,0}=V_1$, then system is operating in the regime of motion (*ass1*). Assume that $V_{s,t} = V_2$. As system starts from the zero state then we have

$r_{1,0} = Q_1, V_{1,0} = V_1; r_{2,0} = Q_1, V_{2,0} = V_1; \ldots, r_{s-1,0} = Q_1, Vs\text{-}1,0 = V_1.$

After time $t^* = (Q^*-Q_1)/(V_1-V_2)$ the particle with number s will change its speed from V_2 to V_1, because

$r_{s,t^*} = Q^* - t^*(V_2 - V_1) = Q_1.$

At the same time we have

$r_{s-1,t^*} = Q_1 + t^*(V_2 - V_1) = Q^* < Q_2.$

Hence, the particle with number (*s-1*) and all other particles will keep speed V_1 and system will operate in the regime of motion (*ass1*).

Necessity

If system starts from the zero state and for some *t* system operates in the regime of motion (*ass1*) then

$r_{1,t} = Q_1, V_{1,t} = V_1; r_{2,t} = Q_1, V_{2,t} = V_1; \ldots, r_{s-1,t} = Q_1, V_{s-1,t} = V_1;$

$V_{s,t}=V_1.$

Thus, it follows $Q_1 \leq r_{s,t} = (1 - (s-1)Q_1) < Q_2$ because $V_{s,t} = V_1$. Hence (18) is held.

Sufficiency

Assume that system starts from the zero state. Moreover, if and (22) is held, then we have

$r_{s,t} = (1 - (s-1)Q_1) > 1-(s-1)Q_2$

As (2.4.19) is held then

$Q_1 < 1 - (s-1)Q_2 < Q_2+Q_1$

for the first particle we have

$r_{1,0} \geq Q_2, V_{1,0} = V_2;$

After time $t^*=(Q_2-Q_1)/(V_2-V_1)$ for the second particle we will have $r_{2,t^*} = Q_2$, $V_{2,t^*} = V_2$. As we consider (*ass2*) with possibly maximal numbers of particles and (2.6.22) is held, then this process will be continued (because distance between particles allows to do it) and finally we will get

$r_{1,t^*}=Q_2, V_{1,t^*}=V_2; r_{2,t^*}=Q_2, V_{2,t^*}=V_2, \ldots, r_{s-1,t^*}=Q_2, V_{s-1,t^*}=V_2;$

$r_{s,t^*} = Q^*$, where $Q_1 < Q^* < Q_2$ and hence $V_{s,t^*} = V_2$,

which means that the system reached the regime (*ass2*). Moreover, the regime (*ass2*), will be reached after the time

$(s-1)(Q_2-Q_1)/(V_2-V_1)$

Necessity

If the system is operating in the (*ass2*) with maximal number of particles then

$r_{1,t} = Q_2$, $V_{1,t} = V_2$; $r_{2,t} = Q_2$, $V_{2,t} = V_2$;..., $r_{s-1,t} = Q_2$,

$V_{s-1,t} = V_2$; $r_{s,t} = Q^*$, and $Q_1 < Q^* < Q_2$.

Hence, (2.4.15) is held.

Sufficiency

If the system starts from initial state as it was mentioned above for (*ass2*), then

$r_{1,0} \geq Q_2$, $V_{1,0} = V_2$;

After the time $t^*=(Q_2-Q_1)/(V_2-V_1)$, for the second particle we will have $r_{2,t^*} = Q_2$, $V_{1,t^*} = V_2$. This process will be continued up to the instant t^{**}, when k_2 particles will have the speed V_2 and for distances we will have

$Q_1 < r_{s,t^{**}} < Q_2$, $V_{s,t^{**}} = V_2$; $r_{s-1,t^{**}} = Q_2$, $V_{s-1,t^{**}} = V_2$,...,

$r_{s-k1,t^{**}} = Q_2$, $V_{s-k1,t^{**}} = V_2$; and

$r_{s-k1-1,t^{**}} = Q_1$, $V_{s-k1-1,t^{**}} = V_1$,..., $r_{2,t^{**}} = Q_1$, $V_{1,t^{**}} = V_1$;

i.e. system is operating in the mixed regime $M(k_1,k_2)$.

Necessity

If system is operating in the regime of motion $M(k_1,k_2)$. Then it is clear that the condition (23) should be held. If system starts from the zero state, then after $t= (Q_2 -Q_1)/(V_2 -V_1)$ *s-th* particle changes speed from V_2 to V_1 because the distance between (*S-1*) and *S-th* particle is becoming equal Q_2. After time $2(Q_2-Q_1)/(V_2-V_1)$ the particle with number (*s-1*) changes speed to V_2. This process will continue while the distance from *s-th* particle to the first particle will be strongly greater than Q_1. Hence, the number of particles, which changed speed from V_1 to V_2 is defined by the equation (24).

Example 2

Let us put $s=30$ and $Q_1=1/30$, $V_1=1$. Then starting from zero state system will reach the regime of motion (*ass1*) and $wtp_{ass1}=1/60=0.0167$.

If $s=20$ and $Q_2=1/20$, $V_2=2$, then starting from the zero state system will reach the regime of motion (*ass2*) and $wtp_{ass2}=1/80=0.0125$.

If $s=12$ and $Q_1=1/30$, $Q_2=1/20$, $V_1=1$, $V_2=2$, then system starting from the zero state will reach M(3,9) and $wtp_{M(3,12)}=0.0129$.

For regime of motion (*sss*) we have $Q_1=1/30$, $Q_2=1/20$, $V_1=1$, $V_2=2$, $e_1 = (1/30)/29=1/870$, $Q_1+e_i=1/29$, $wtp_{sss}= 0,009$.

The condition $Q_1/V_1 < Q_2/V_2$ is held ($1/30<1/20$). Simple calculations yield $wtp_{sss}=0.009 < wtp_{ass2}=0.0125 < wtp_{M(3,12)} = 0.0129 < wtp_{ass1}=0.0167$.

Comparison of various regimes of motion gives gain in an average waiting time of particle in some fixed point of a circle:

(*sss*) with (*ass2*) is 25%; (*sss*) with *M(3,12)* is 30%; (*sss*) with (*ass1*) is 46% (*ass2*) with *M(3,12)* is 3.2%; (*ass2*) with (*ass1*) is 26%; *M(3,12)* with (*ass1*) is 22%.

Maximum gain with using (*sss*) regime of motion in comparison with other regimes is 44%, but comparison of (*ass2*) with (*ass1*) regime shows advantage of the regime (*ass2*). If V_2 takes higher value, then the gain in average waiting time can be higher. The following example shows these facts.

Example 3

Let us put $s=30$ and $Q_1=1/30$, $V_1=1$. Then starting from zero state system reaches the regime of motion (*ass1*) and $wtp_{ass1}=1/60=0.0167$.

If $s=20$ and $Q_2=1/20$, $V_2=4$, then starting from the zero state system reaches the regime of motion (*ass2*) and $wtp_{ass2}=1/160=0.0062$.

If $s=22$ and $Q_1=1/30$, $Q_2=1/20$, $V_1=1$, $V_2=4$, then system starting from the zero state reaches $M(6,16)$ and $wtp_{M(6,16)}=0.0088$.

For regime of motion (*sss*) we have $Q_1=1/30$, $Q_2=1/20$, $V_1=1$, $V_2=4$, $e_i = (1/30)/29=1/870$, $Q_1+e_i=1/29$, $wtp_{sss}= 0,0045$.

The condition $Q_1/V_1 < Q_2/V_2$ is held ($1/30<1/20$). Simple calculations yield $wtp_{sss} = 0.0045 < wtp_{ass2} = 0.0062 < wtp_{M(6,16)}=0.0088 < wtp_{ass1}=0.016$.

Comparison of various regimes of motion gives gain in an average waiting time:

(*sss*) with (*ass1*) is 72%; (*sss*) with $M(6,16)$ is 49%; (*sss*) with (*ass2*) is 46%; (*ass2*) with $M(3,12)$ is 29%; (*ass2*) with (*ass1*) is 61%; $M(6,16)$ with (*ass1*) is 45%;

Thus, if V_2/V_1 is going up then the gain also goes up.

2.6.2 Stochastic Systems. Introduction of delays

We will introduce for each system stochastic delays, i.e. at some random instants the speed of some particle, which moves with the (high) speed V_2 will be changed to the (low) speed V_1. This is a typical situation for applications, because in traffic systems vehicles (buses, trains, cars) can face with some unexpected barriers (appearance of animals in the highways, sometimes a quality of the road does not allow to have high speed and so on). In communication systems the speed of information getting can be reduced due to broken or reliability of the equipment.

Definition 2.6.5

If in some instant t a particle having the speed V_2 must change its speed from V_2 to V_1, then we say that delay happened in the system.

Definition 2.6.6

The regime of motion is called to be *invariant* to the delays (*id*), if after occurring of delay the system can come back to the regime, where it was at the preceding instant of the delay.

Theorem 2.6.3

For constructed mathematical models the following statements are true:

i. The regimes of motion (*ass2*) and (*mss*) are invariant to the delay
ii. The delay transforms the regime of motion (*sss*) into (*ass1*).

Proof. (i) - part.

If delay occurs in the regime of motion (*ass2*), then one particle, let us say *i-th*, changes its speed from V_2 to V_1. As we consider only the regimes, which can be reached starting from the zero position, then according to construction of the mathematical model there can be three options:

a. The distance between *i-1*, *i-th* and (*i+1*) particles equal Q_2, i.e. $r_{i\text{-}1,t} = r_{i,t} = Q_2$;
b. The distance between i-th and (i+1) particle
   $Q_1 < r_{i,t} < Q_2 + Q_1$ and $r_{i\text{-}1,t} = Q_2$
c. The distance between (*i-1*)-*th* and *i-th* particle
   $r_{i\text{-}1,t} < Q_2 + Q_1$ and $r_{i,t} = Q_2$
a. part. Denote t^* a duration of the delay. As $V_{i,t} = V_1$, then
   if $t^* < (Q_2 - Q_1)/(V_2 - V_1)$, after finishing the delay we have:

$r_{i,t^*} = Q_2 + t^*(V_2 - V_1) > Q_2$, $V_{i,t^*} = V_2$, $r_{i\text{-}1,t^*} = Q_2 - t^*[(Q_2 - Q_1)/(V_2 - V_1)](V_2 - V_1) > Q_1$, $V_{i\text{-}1,t^*} = V_2$

Hence, $V_{i\text{-}1,t} = V_2$ and system operates in the regime of motion (*ass2*).

If $t^* \geq (Q_2 - Q_1)/(V_2 - V_1)$, then after finishing the delay we have:

$r_{i,t^*} = Q_2 + t^*(V_2 - V_1) > Q_2$, $V_{i,t^*} = V_2$,

$r_{i\text{-}1,t^*} = Q_2 - t^*[(Q_2 - Q_1)/(V_2 - V_1)](V_2 - V_1) = Q_1$, $V_{i\text{-}1,t^*} = V_1$.

If a duration of delay is greater or equal $(s\text{-}1)(Q_2 - Q_1)/(V_2 - V_1)$ then system passes into the zero state and hence after some time according to the theorem 2.6.1 will come to the regime of motion (*ass2*). Parts (b) and (c) are proved the same way.

Similarly, the statement of the theorem 2.6.3 is proved for (*mss*) regime.

part. Consider the regime of motion (*sss*). Assume that *i-th* particle changes its speed from V_2 to V_1. This (*i-th*) particle never can change the speed again to V_2, because for any t is held $r_{i,t} < Q_2$. After time ei $/(V_2 - V_1)$ the (*i-1*) changes speed to V_1 and this process will be continued until the time when all particles will get the speed V_1. Thus, the system transforms into the regime (*ass1*). As the condition of the theorem 2.6.1 are held, because system starts its operation from the regime (*ass1*), hence in stationary regime (after time $(e_1 + e_2 + ... + e_i)/(V_2 - V_1)$) system will operate in (*ass1*).

Hence, $V_{i\text{-}1,t} = V_2$ and system operates in the regime of motion (*ass2*).

Thus, according to the theorem 2.6.3, if in saturated regime *just only once* one particle will change speed from V_2 to V_1, then this particle should keep this the low speed V_1, because for getting the high speed V_2 the distance to the following particle should be equal Q_2 but it is not possible, because according to the construction of the saturated regime a distance between neighbor particles is always less than Q_2.

It is necessary to denote that introducing of the delays for saturated regime force the system transforms from this regime into another regime of motion in our case into the regime (*ass1*). Moreover, the system has no possibility to leave (*ass1*), for instance goes back to (*sss*) or in another regime. Hence, if in the system it is expected an occurrence of any delays, then it is not recommended to have regime of motion (*sss*).

In this meaning, if the systems are faced with delays, then an optimal regime is not (*sss*). Hence, the regime of motion (*sss*) is not recommended for applications, because in the practice there always exist delays.

Let us introduce the following additional conditions.

At some instant t there occurs some delay in the system, i.e. the speed of one particle (does not matter which one), which is moving with the speed V_2 is changed to the speed V_1 (*i*)

We call it an occurrence of a delay.

Theorem 2.6.3

Under the condition (i) the following expression is held

$wtp_{ass2} \leq wtp_{M(k1,k2)} \leq wtp_{ass1} = wtp_{sss}$

Proof

Assume that in the regime of motion (*sss*) at the instant t^* *i-th* particle will change speed from V_2 to V_1, i.e.

$r_{1,t^*} = Q_1+e_1,\ V_{1,t} = V_2;\ rr_{2,t^*} = Q_1+e_2,\ V_{2,t} = V_2;\ldots,$

$r_{i-1,t^*} = Q_1+e_{i-1},\ V_{i-1,t^*} = V_2;\ r_{i,t^*} = Q_1+e_i,\ V_{i,t^*} = V_1,\ e_i>0,$

$r_{i+1,t} = Q_1+e_{i+1},\ V_{1,t} = V_2;\ldots.\ r_{s,t^*} = Q_1+e_s,\ V_{s,t^*} = V_2;$

Then after the time interval $e_i/(V_2-V_1)$ i.e. at the instant $t^{**}=t^*+e_{i-1}/(V_2-V_1)$ the distance $r_{i-1,t} = Q_1$ and hence $V_{i-1}=V_1$. Step by step the distances between particles become Q_1. This process will be continued until the instant when all particles will have speed V_1 and hence the system passes into the regime of motion (*ass1*).

If in the regime of motion (*ass2*) at the instant t^* occurs a delay (speed of *i-th* particle is changed from V_2 to V_1) then we have

$r1,t^* = Q_2,\ V_{1,t^*} = V_2;\ r_{2,t^*} = Q_2,\ V_{2,t^*} = V_2;\ldots,$

$r_{i-1,t^*} = Q_2,\ V_{i-1,t^*} = V_2;\ r_{i,t^*} = Q_2,\ V_{i,t^*} = V_1,$

$r_{i+1,t^*} = Q_2,\ V_{i+1,t^*} = V_2;\ldots.\ r_{s,t^*} = Q_2,\ V_{s,t^*} = V_2;$

The long delay (occurs only once) can lead the system to the zero position i.e. there exists some t^* for which we have

$r_{1,t^*} = Q_1,\ V_{1,t^*} = V_1;\ r_{2,t^*} = Q_1,\ V_{2,t^*} = V_1;\ldots,$

$r_{s-1,t^*} = Q_1,\ V_{s-1,t^*} = V_1;\ r_{s,t^*} \geq Q_2,\ V_{s,t^*} = V_2,$

As the system came from the regime of motion (*ass2*) then the expression (2.6.19) is held, i.e.

$1-(Q_2-Q_1)/Q_1 \leq s < 1/Q_1$

and using the statement of Theorem 2 (all conditions of Theorem 2 are held) we are getting that the system will come to the regime of motion (*ass2*).

Conclusion

In the paper mathematical models of moving particles on a circle describing behavior of transport systems are constructed. In the capacity of an efficiency index of the system an average waiting time of particle in some point of the circle is taken. The diagram allowing to define various states of the system. It is obtained the effect when increasing of moving particles (transport units) can worse effectivity of operation. The method for finding an optimal number of particles minimizing the efficiency index of the system. Numerical examples demonstrating theoretical results are given.

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