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Moment Distribution Analysis of Frames with Tapered Members

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Abstract

The traditional moment distribution method is modified to analyze statically indeterminate beams and frames that consist of prismatic and tapered members. An expression of stiffness and carry over factors for tapered members is derived based on the solution of a second order differential equation for the curvature of a rectangular cross-sectional shape tapered in depth only.

The moment distribution method is based on distributing the applied moment at any joint to the members' ends that connected to this joint. The distributed moment can be mathematically obtained from multiplying the applied moment by a distribution factor. Meanwhile, the distributed moment is carried over to the far end of the member based on a carry-over factor of 0.5. In the modified method there are two different carry over factors; one of them is less than 0.5 and the other is larger, depending on the tapering ratio. Three different applications on the portal frame with different support conditions and different columns' shapes are presented with numerical analysis and results.

Keywords: Distribution moment; Stiffness factor; Carry over factor; Prismatic member; Tapered member

Symbols

d_1 and d_2: depth at end 1 and end 2 respectively,
I_1, I_2: moment of inertia at end 1and end 2 respectively,
k: stiffness of member,
y: deflection,
u: tapering ratio,
C.O.F.: carry over factor,
C.O.M.: carry over moment,
D. M: distribution factor,
E: Young's modulus,
F.E.M.: fixed end moment,
L: member length,
M_1, M_2: bending moment at end 1 and end 2 of the member, and
θ_1, θ_2: rotation at end 1 and 2 respectively.

Introduction

The moment distribution method is one of main analysis methods that used in the analysis of indeterminate structures with prismatic members. Hardy Cross in 1932 developed a new method that can be used to analyze (Lightfoot 1961); a frame consists of prismatic beams and columns. In this study, a modified moment distribution method is derived to analyze the indeterminate structures with prismatic and /or prismatic members.

The tapered member under consideration has a rectangular cross section tapered in depth only with dimensions and tapering ratio as shown in Figure (1).

The moment distribution is based on the distributed of unbalanced moment to the near ends of the connected members according to their distribution factors, assuming that all far ends of these members are fixed. Meanwhile, moment that carried over to the far end of each member is equal the carry over factor times the moment of the near end (Wang 1957 and Norris 1960).

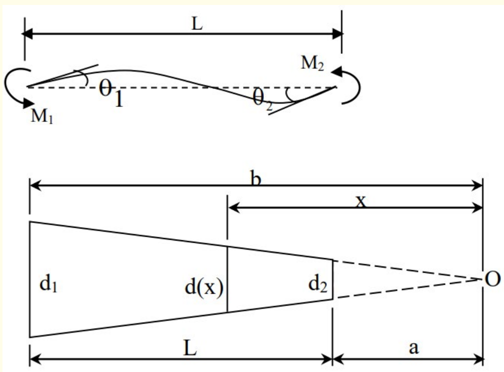

Figure 1: Model shape.

Mathematical Expression of Tapered Member Stiffness

The mathematical expression for stiffness of tapered member is derived on the solution of a second order differential equation that given below (Al-Sarraf, 1979 and Al-Sarraf and Yossif, 2005) which represents the curvature of a tapered member subjected to moments at ends.

$$EI(x)\frac{d^2y}{dx^2} = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \qquad (1)$$

Moment of inertia of the tapered member is defined as:

$$I(x) = \left(\frac{x}{a}\right)^3 I_2 \qquad (2)$$

where

I_1, I_2: moment of inertia at end 1 and end 2 respectively,
a: distance from origin O to the end 2 of member, and
b: distance from origin O to the end 1 of member.
Where b/a = u

The second order differential equation that given in eq.(3) can be obtained by substituting eq.(2) in to eq.(1).

$$EI_2\left(\frac{x}{a}\right)^3\frac{d^2y}{dx^2}=\frac{M_1}{L}(x-a)+\frac{M_2}{L}(x-b) \qquad (3)$$

This equation can be simplified to get eq.(4).

$$EI_2\frac{d^2y}{dx^2}=\frac{a^3}{L}\left[(M_1+M_2)x^{-2}-a(M_1+uM_2)x^{-3}\right] \qquad (4)$$

The first integration of eq.(4) with respect to x gives eq.(5).

$$EI_2\frac{dy}{dx}=\frac{a^3}{L}\left[-(M_1+M_2)x^{-1}+0.5a(M_1+uM_2)x^{-2}\right]+A \qquad (5)$$

Second integration of eq.(5) gives eq.(6),

$$EI_2 y=\frac{a^3}{L}\left[-(M_1+M_2)\ln x-0.5a(M_1+uM_2)x^{-1}\right]+Ax+B \qquad (6)$$

The unknowns A and B are found in eq. (7) and eq. (8) after substituting boundary conditions at the end of member in eq. (5) and eq.(6). Where the deflection y equal zero at member ends when x = a and x =b.

$$A=\frac{5a}{(u-1)^2EI_2u}\left[M_1\left(1-u-2u.\ln\left(\frac{1}{u}\right)\right)+u.M_2\left(1-u-2\ln\left(\frac{1}{u}\right)\right)\right] \qquad (7)$$

$$B=\frac{5a^2}{(u-1)^2EI_2u}\left[M_1\left(u^2-1-2u(\ln(b)-\ln(a).u)\right)+u.M_2\left(u^2-1-2(\ln(b)-\ln(a).u)\right)\right] \qquad (8)$$

The unknowns M_1 and M_2 can be obtained by substituting the boundary conditions at member ends in eq.(5). Where the rotations dy/dx equal θ_2 at x = a and equal θ_1 at x = b. The equations of M_1 and M_2 arranged in a matrix form as given in eq.(9).

$$\begin{Bmatrix}M_1\\M_2\end{Bmatrix}=\frac{EI_2}{L}\begin{bmatrix}\psi_1 & \psi_0\\ \psi_0 & \psi_2\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} \qquad (9)$$

Where:

d_1, d_2: depth at end 1 and end 2 respectively for non-prismatic member,
u: tapering ratio that equal b/a
E: Young's modulus,
L: member length,
M_1, M_2: bending moments at end 1 and 2 respectively for non-prismatic member,
θ_1, θ_2: rotations at end 1 and 2 of member due to bending moment, and
ψ_0, ψ_1, ψ_2: stiffness factor that defined in eq. (10), eq. (11) and eq, (12), respectively.

$$\psi_1=u^2\left[\frac{2\ln u+(u-3)(u-1)}{(u+1)\ln u-2(u-1)}\right] \qquad (10)$$

$$\psi_0=u\left[\frac{u^2-1-2u\ln u}{(u+1)\ln u-2(u-1)}\right] \qquad (11)$$

$$\psi_2 = \frac{2u^2 \ln u - \left(3u^2 - 4u + 1\right)}{\left(u+1\right)\ln u - 2\left(u-1\right)} \qquad (12)$$

The stiffness factor that obtained from eq.(10), (11) and (12) can be used to find the carry over factor, which are the ratios of moments transferred from the near end of member to the far end in the moment distribution method.

Carry Over Factor

The carry-over factor is defined, as the end moment due to a unit rotation at the other end of the member while the end is fixed. The carry-over moment that is induced at the far end can be computed by multiplying the carry-over factor by the moments at the near end. The general carry- over factor for prismatic and tapered members can be obtained from eq. (13) when the moment transferred from the near end (with the larger depth) to the far end of member (with the smaller depth), and from eq. (14) when the moment transferred from the near end (with the smaller depth) to the far end of member (with the larger depth).

$$C.O.F_1 = \frac{\psi_0}{\psi_1} \qquad (13)$$

$$C.O.F_2 = \frac{\psi_0}{\psi_2} \qquad (14)$$

where:

$C.O.F_1$: Carry-over factor of moments from end 1 to end 2, and
$C.O.F_2$: Carry-over factor of moments from end 2 to end 1

The carry over factors of prismatic member that equal 0.5 (Livesly, 1956) is obtained from both eq. (13) and eq. (14). Mathematically the values of these two equations are indeterminate due to numerators and denominators are both zero when substituting tapering ratio u=1 (for prismatic shape only), therefore the value of these equation can be obtained using stronger form of l'Hopital's rule (which is depend on finding the derivatives of both numerators and denominators separately three times to get the determinate results) (Finny and Thomas, 1990).

For a member of varying rigidity (EI), the carry-over factors of tapered shape can be obtained using eq. (13) and eq. (14) according to the larger and smaller depths. The numerical values of the carry over factors is tabulated in Table (1) for tapering ratio ranged between u=1 to u=5.

Fixed End Moments

The fixed end moments acting at the end of a member due to the applied loads on the member are denoted by symbol F.E.M. These moments distributed between member ends according to the stiffness factor of members at connected joints. The fixed end moments can be determined by the method of consistent deformation for straight prismatic and non- prismatic members. The values of fixed end moments for a member subjected to a uniform distributed load can be obtained from eq.(15) for prismatic members. On the other hand, for non-prismatic members with tapering ratio equal to 1.5, eq.(16) and (17) can be used.

$$F.E.M. = 0.083\ w.L^2 \quad (15)$$

$$F.E.M._{(\text{for small depth})} = 0.061\ w.L^2 \quad (16)$$

$$F.E.M._{(\text{for larg depth})} = 0.107\ w.L^2 \quad (17)$$

u	Stiffness factors			Carry over factors	
	ψ_1	ψ_0	ψ_2	ψ_0/ψ_1	ψ_0/ψ_2
1	4.000	2.000	4.000	0.500	0.500
1.1	4.959	2.308	4.298	0.465	0.537
1.2	6.038	2.631	4.593	0.436	0.573
1.3	7.240	2.970	4.884	0.410	0.608
1.4	8.571	3.322	5.173	0.388	0.642
1.5	10.034	3.689	5.460	0.368	0.676
1.6	11.632	4.070	5.744	0.350	0.709
1.7	13.369	4.464	6.026	0.334	0.741
1.8	15.249	4.872	6.307	0.319	0.772
1.9	17.275	5.292	6.585	0.306	0.804
2	19.451	5.725	6.863	0.294	0.834
2.1	21.779	6.171	7.138	0.283	0.864
2.2	24.263	6.629	7.413	0.273	0.894
2.3	26.906	7.098	7.686	0.264	0.924
2.4	29.712	7.580	7.958	0.255	0.952
2.5	32.684	8.074	8.229	0.247	0.981
2.6	35.824	8.579	8.499	0.239	1.009
2.7	39.136	9.095	8.768	0.232	1.037
2.8	42.623	9.623	9.037	0.226	1.065
2.9	46.288	10.161	9.304	0.220	1.092
3	50.133	10.711	9.570	0.214	1.119
3.1	54.162	11.272	9.836	0.208	1.146
3.2	58.378	11.843	10.101	0.203	1.172
3.3	62.783	12.425	10.365	0.198	1.199
3.4	67.380	13.018	10.629	0.193	1.225
3.5	72.173	13.621	10.892	0.189	1.251
3.6	77.163	14.234	11.154	0.184	1.276
3.7	82.353	14.858	11.416	0.180	1.302
3.8	87.748	15.491	11.677	0.177	1.327
3.9	93.348	16.135	11.937	0.173	1.352
4	99.156	16.789	12.197	0.169	1.376
4.1	105.177	17.453	12.457	0.166	1.401
4.2	111.411	18.126	12.716	0.163	1.426
4.3	117.862	18.810	12.974	0.160	1.450
4.4	124.532	19.503	13.232	0.157	1.474
4.5	131.424	20.205	13.490	0.154	1.498
4.6	138.541	20.918	13.747	0.151	1.522
4.7	145.885	21.639	14.004	0.148	1.545
4.8	153.458	22.370	14.260	0.146	1.569

4.9	161.263	23.111	14.516	0.143	1.592
5	169.303	23.861	14.772	0.141	1.615

Table 1: Carry over factors.

Applications

Three different applications are presented below using same portal frame shapes, dimensions, material and loading but have different column shapes and support conditions. The moment distribution method is used to analyze these models under the effects of moment due to the uniform distributed loads.

Application 1: A steel portal frames loaded with uniformly distributed load equal to 40 kN/m as shown in Figure (2), fixed support at base of columns and supported against side sway. The modulus of elasticity equal to $2x10^8$ kN/m^2. The frame consist of one prismatic beam of 6 m length with rectangular cross sectional dimensions of 0.06 m depth and 0.025 m width and two prismatic columns have the same dimensions of the beam.

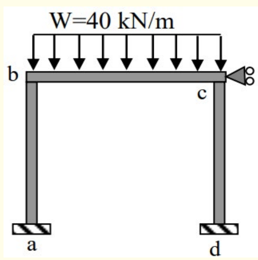

Figure 2: Portal frame for application one.

Analysis: The distribution factors at joints b and c equal 0.5 due to the stiffness of member (bc) equal to the stiffness of member (ba) and (cd). The carry over factor for prismatic members is 0.5 (Table (1)). The fixed end moment due to uniformly distributed load on beam equal to 120 kN-m using eq. (15).

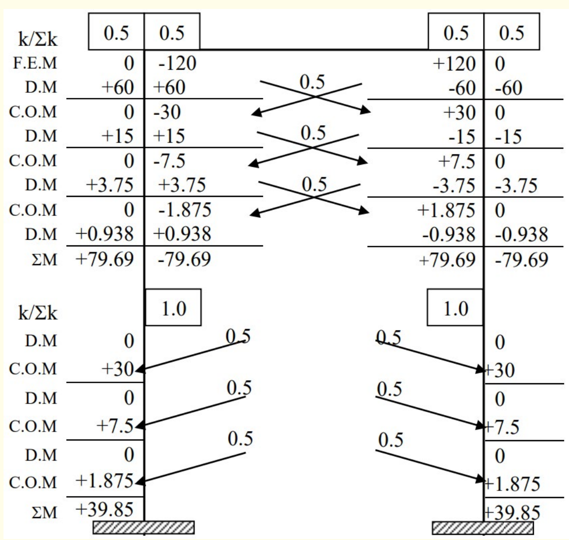

Figure 3: Distribution moments for Application 1.

Application 2: A steel portal frames loaded with uniformly distributed load equal to 40 kN/m as shown in Figure (4), fixed support at base of columns and support against side sway. The modulus of elasticity equal to $2x10^8$ kN/m^2. The frame consists of one prismatic beam of 6 m length with rectangular cross-sectional dimensions of 0.06 m depth and 0.025 m width and two non-prismatic columns have tapering ratio of 1.5. The length of each column is 6 m and cross-sectional dimensions equal to 0.06 m depth and 0.025 m width at smaller end depth, which located at the base of columns.

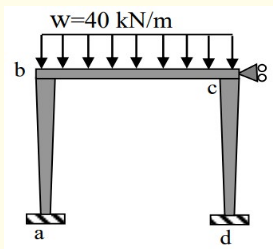

Figure 4: Portal frame for application two.

Analysis: The modulus of elasticity, moment of inertia and length are equals for members ab, bc and cd, while the stiffness equation of each member is equal to $EI_2\psi_1/L$ at larger member depth and equal to $EI_2\psi_2/L$ at smaller member depth. The stiffness factor depends on ψ_1 only at joint b and c. Where ψ_1 = 4 for member bc and ψ_1 = 10.034 for member ba (from Table (1) at u=1.5), therefore the stiffness factor of member ba=0.715 and for member bc=0.285. The carry over factor for prismatic member equals to 0.5 while the carry over factor for tapered columns equals to 0.368 (from larger to smaller depth) and 0.676 (from smaller to larger depth) from Table (1) at tapering ratio of 1.5. The fixed end moment is equal to 120 kN-m using eq. (15), (due to uniformly distributed load on the prismatic member). The analysis results are shown in Figure (5) and Table (2).

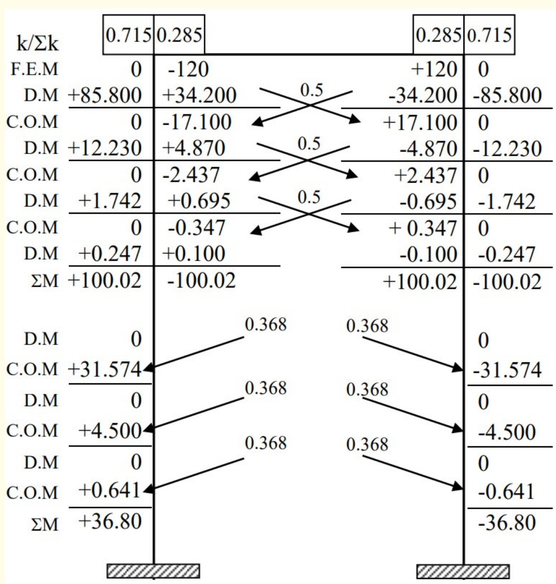

Figure 5: Distribution moments for Application 2.

Moments value kN-m		***Joint a***	***Joint b***	***Joint c***	***Joint d***
Suggested method		36.80	100.02	100.02	36.80
STAAD III results	6 members	36.752	108.776	108.776	36.752
	12 members	36.766	103.827	103.827	36.766
	18 members	36.779	101.919	101.919	36.779
	24 members	36.780	100.750	100.750	36.780
	30 members	36.792	100.046	100.046	36.792

Table 2: Moment Distribution Results for application two.

Application 3: A steel portal frames loaded with uniformly distributed load equal to 40 kN/m as shown in Figure (6), hinge support at base of columns and supported against side sway. The modulus of elasticity equal to $2x10^8$ kN/m^2. The frame consists of one prismatic beam of 6 m length with rectangular cross sectional dimensions of 0.06 m depth and 0.025 m width and two non-prismatic columns have tapering ratio of 2. The length of each column is 6 m and cross sectional dimensions equal to 0.06 m depth and 0.025 m width at smaller end depth, which located at the base of columns.

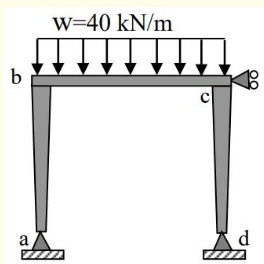

Figure 6: Portal frame for application three.

Analysis: The modulus of elasticity, moment of inertia and length are equals for members ab, bc and cd, while the stiffness equation of each member is equal to $EI_2\psi_1/L$ at larger member depth and equal to $EI_2\psi_2/L$ at smaller member depth. The stiffness factor depends on ψ_1 only at joint b and c. Where $\psi_1 = 4$ for member bc and $\psi_1 = 19.451$ for member ba (from Table (1) at u=2), therefore the stiffness factor of member ba=0.83 and for member bc=0.17. The carry over factor for prismatic member equals to 0.5 while the carry over factor for tapered columns equals to 0.294 (from larger to smaller depth) and 0.834 (from smaller to larger depth) from Table (1) for tapering ratio of 2. The fixed end moment is equal to 120 kN-m using eq. (15), (due to uniformly distributed load that subjected on the prismatic member). The analysis results are shown in Figure (7) and Table (3).

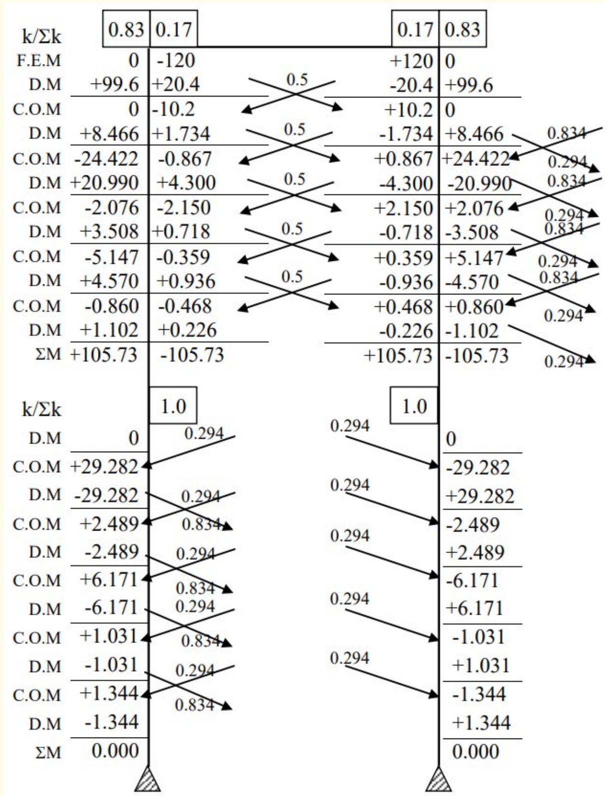

Figure 7: Distribution moments for Application 3.

Moments value kN-m		***Joint a***	***Joint b***	***Joint c***	***Joint d***
Suggested method		0	105.73	105.73	0
STAAD III results	6 members	0	105.453	105.453	0
	12 members	0	105.464	105.464	0
	18 members	0	105.475	105.475	0
	24 members	0	105.487	105.487	0
	30 members	0	105.596	105.596	0

Table 3: Moment Distribution Results for application three.

Conclusions

The moment distribution method is used in this research to analyze frame includes prismatic and non-prismatic members shape. Carry over factors for non-prismatic members have been derived in this study to show difference in values according to the tapering ratio. This carry over factor can be classified into two types; the first is the carry over factor of moment distribution from larger to the smaller member depth while the second is the carry over factor of moment distribution from smaller to the larger member depth, which are equal value in the case of prismatic members.

The main recommendation of this work is that the moment distribution method can be used in the analysis of frames included tapered members by replacing the common carry over factor (0.5) by another suitable value depending on the tapering ratio. Three different applications are presented in this work solved by hand calculation by suggested method and the finite element method using two software programs. The analysis results from the finite element method are convergence to results from the suggested method when using a large number of equivalent members, in which the suggested method gives an exact numerical solution.

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