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The AltiMetric Workspace

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Abstract

In essence, this article introduces a digital platform (see Figure 1) that can be located on the Earth's surface, or anywhere above or below it. It provides a local project "workspace" that can be tied precisely to the Earth-Centered Earth-Fixed (ECEF) coordinate system of an ellipsoid. It is, in nature, like the old Polar Projection, with three differences:

1. It works anywhere on Earth.
2. It works in space (above or below ground), or other celestial bodies.
3. It can hold three-dimensional projects.

It is locked in space by one desired reference point and is designed to always be parallel to the ellipsoid at that point. Note that being tangent or secant to an ellipsoid is not a necessary condition. This workspace is named "AltiMetric".

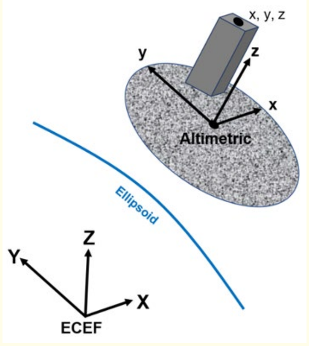

Figure 1

Engineers operate in Cartesian workspaces (X, Y, Z spaces), while GIS databases normally are expressed in geodetic workspaces (projections, or some X, Y spaces).

This proposed solution resembles the mentioned "Polar Projection" that consists of simply attaching a plane to an ellipsoid. The proposal herein uses such a plane for local projects but places it at any required elevation (or depth). This solution, as formulated, creates a strong link between such a *three-dimensional* CAD space with its geodetic *three-dimensional* position.

A proposed solution

This approach guarantees that any local workspace thus created will result in coordinates easily converted to any of the existing geodetic or State Plane georeferences. This approach utilizes the simplest type of all the projection types available, normally identified as "Polar Projection", which consists of a plane that is placed around some convenient "anchor" point on the ellipsoid.

However, there is a major difference with that projection: The Polar Projection is tangent to the ellipsoid, while the solution proposed herein uses a plane that is located at any elevation or depth, on or away from the ellipsoid. Therefore, the purpose of this paper is not to claim a new type of projection, but to introduce a method that can be used to easily convert CAD data into this framework. See Figure 2. It shows an ellipsoid "**E**" and a plane "**A**" that goes through point **C** and is parallel to the ellipsoid at **C**.

This surface **A** is at an elevation that is suitable for local data (possibly an average elevation of a project) and is parallel with the local ellipsoid at some point **C** of choice.

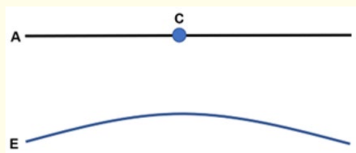

Figure 2

This surface **A** is at the core of this proposal. A can be located anywhere, and it does not have to be tangent or secant to an ellipsoid, like a projection is. It does not even have to be near ground level. For example, in Denver, one could use and elevation of 1,609 meters ASL (mile high) to hold that plane in place.

This plane is held by a central anchor point **C** having specific and known ECEF coordinates. Therefore, **C** is also fixed by its geodetic coordinates (latitude, longitude and ellipsoidal heigh) which are easily converted to the Colorado State Plane Coordinate System with existing software. Point **C** can also be chosen at will.

Also, within that plane, there is a local x, y, z Cartesian and metric coordinate system with a central origin at **C**, where the local coordinates **x**, **y**, and **z** are equal to 0, 0, 0 meters.

The **x** and **y** axes lie within plane **A**. The **y** axis always points toward Astronomic North. The **z** axis is perpendicular to (a) the ellipsoid at point C and (b) to plane **A**. Any nearby geodetic point that has a known latitude, longitude and geoidal height can be converted to **x**, **y**, **z** AltiMetric coordinates within that grid.

All established geodetic constructs (ellipsoids, projections, etc.) remain in place. **x**, **y**, **z** positions can be converted to conventional geodetic and geocentric coordinates (latitude, longitude, ellipsoidal height, **X**, **Y**, **Z**, etc.).

At present, legacy projections, as defined, provide a local coordinate system consisting of only a two-dimensional reference, and no vertical coordinate (above or below the projection plane) is calculated. In the solution herein, a local **z** coordinate is also calculated (height above or below the **A** plane), making it a three-dimensional solution. This opens the door to introducing engineering designs into GIS databases.

This proposed plane then represents a local Cartesian *workspace* that can be used at any elevation (alti) and is metric. Hence the name "**AltiMetric Workspace**", or "**AM**".

Description of the new workspace

AM is a local plane that has the following characteristics:

AM is a *local Cartesian workspace* that is located at some convenient elevation above the ellipsoid surface, as set by ellipsoidal height using the elevation of a chosen project point. For example, it may fit an average elevation of the local ground surface that may have an elevation many meters above or below the sea level (i.e., Denver, the Altiplano of Peru, the Mariana Trench, etc.).

AM is metric (like the Earth Centered Earth Fixed, or ECEF system).

AM is fixed in place by a central "origin", called point "C". Its position is given by:

1. A geodetic latitude (**φC**).
2. A geodetic longitude (**λC**).
3. A geodetic ellipsoidal height (**ehC**).
4. Local Cartesian coordinates **x=0**, **y=0**, **z=0** in [meters]; and
5. Earth-Centered, Earth-Fixed (ECEF) coordinates **X**, **Y**, **Z**.

AM is based on the Polar Projection. It uses elements of geodetic mathematics, combined with elements of the Helmert (orthogonal) transformation as part of the solution:

- It is neither necessarily tangent nor secant to the ellipsoid.
- The **AM** plane is perpendicular to the vertical to the ellipsoidal height at point **C**.
- It is not based on a projection surface that needs to be "developed" to produce a plane.
- Any data distortions that are due to Earth not being flat are deemed negligible in a local sense.

By design, the **AM** plane is parallel to the ellipsoid (any desired datum) at point **C**. Therefore, the **AM** plane is inclined with respect to the Equator (as defined by the ECEF) by an angle "i", as presented in Appendix 1 to this document, as deducted by taking the first differential of the equation for the ellipsoidal Meridian going through C.

For identification, each **AM** is named after the location of its origin **C**, i.e., a town's name, a project name, etc.

Each **AM** is strictly a local solution. The extent is selected by the user according to acceptable error levels across a project area. This possible error is quantified in this paper.

There is no definition of zones, but they may be constructed with separate **AM** planes if needed.

AM is a three-dimensional workspace (**x**, **y**, **z**). The **z** coordinate within AM is a height above that AM plane, as inherited from a CAD file, or a real elevation (orthometric height), as inherited from local map data. However, for many uses, like in outer space, the local z coordinate of AM can be used instead of elevation. Besides, an elevation within AM can be refined from the z coordinate by applying two corrections: One for earth curvature and another for geoidal height.

Therefore, to bring CAD data into AM, one must do the following:

- One identifies a suitable point **C** and obtains its horizontal and vertical geodetic position (Latitude, Longitude, Orthometric Height).
- Then one identifies three or more points surrounding **C** and obtains their **AM** position from the CAD file. These are the horizontal control data for the transformation.

- The rotations are defined as shown below.
- Then one obtains the geodetic elevations of these points. This completes the data needed for the three-dimensional transformation.

The **AM** is constructed from existing geodetic quantities and formulae, allowing the conversion from one reference to another (geodetic <> ECEF <> **AM** <> ECEF <> geodetic). This means that **AM** is really a sort of missing link between geodetic coordinates and local reference systems. Figure 3 (on the next page) shows the current/existing conversion workflow within geodesy, while Figure 4 shows how AM is inserted into that standard process (*bold boxes*).

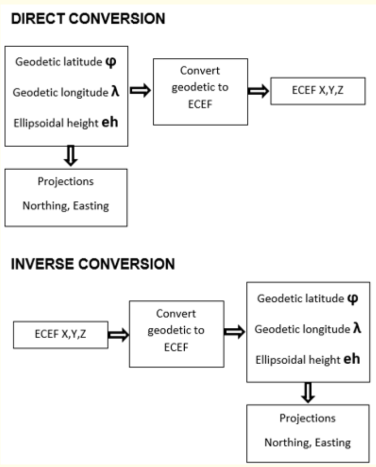

Figure 3: The standard geodetic process.

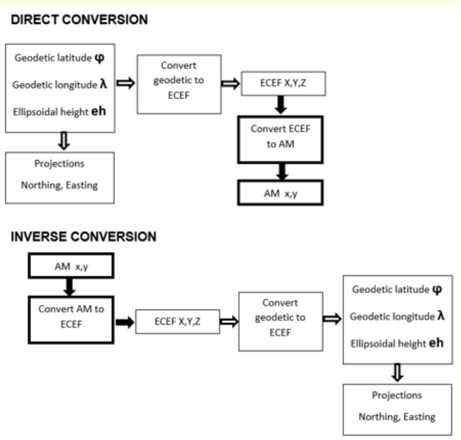

Figure 4: Addition of AM to the conversion workflow.

Establishment of the AM workspace

This enhancement is achieved by first creating this AM workspace once and permanently for a location, and then the use of that local workspace. Please refer to Figure 5-a that shows the AM plane (grey), with its Cartesian grid x, y, z. A complete list of the minimum necessary parameters to this are the following:

- **X**, **Y**, **Z** is the ECEF system. Its X-axis points to Greenwich. The **Z** axis is the ellipsoid's axis of rotation.
- **AM** is the Altimetric Plane (grey disk), with its local **x**, **y**, **z** Cartesian grid.
- **Point D** is located on the ellipsoid (ellipsoid not shown) and has ECEF/geocentric coordinates $\mathbf{X_D}$, $\mathbf{Y_D}$, $\mathbf{Z_D}$. This point D is set on the ellipsoid as the first step in determining the location of the AM system, and its ECEF coordinates are calculated from the corresponding latitude and longitude position.
- **Point C** is above point **D**, a distance **eh**, which is the ellipsoidal height of **C** and is perpendicular to the ellipsoid at **D**.
- **AM** remains parallel to the ellipsoid surface at **C**.
- **AM** axis **z** and **eh** are perpendicular to the ellipsoid and **AM**.
- Angle "**i**" is the inclination angle of the meridian at D. It is therefore the inclination of AM with respect to the Equatorial plane (X, Y plane). It also is the inclination of line D-C with respect to the Z axis (for an explanation of its value please see "Appendix 1 - Altimetric Plane Inclination" at the end of this document).
- The distance from **O** to **D** (not shown) we will call "**R**". It is not perpendicular to anything. It is calculated using Pythagoras based on the differences of the ECEF coordinates of points O and D.

In other words, all the above parameters belong to the position and orientation of the AM plane and remain the same once it is selected for a specific location **C**.

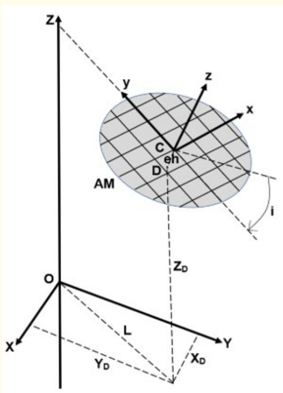

Figure 5-a

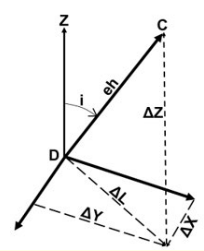

Figure 5-b

Figure 5-b shows detail between points D and C:

$\mathbf{X_D}$, $\mathbf{Y_D}$, $\mathbf{Z_D}$ are ECEF coordinates of a selected point **D** (on the ellipsoid). They are calculated from latitude and longitude using the standard geodetic conversion formulae (not provided in this paper).

$L = \text{sqrt}\ (X_D\text{^}2 + Y_D\text{^}2)$
$R = \text{sqrt}\ (L\text{^}2 + Z_D\text{^}2)$
$\Delta L = eh * \text{sine}(-i)$
$\Delta X = X_D * \Delta L/L$ or $\Delta X = X_D * eh * \text{sine}(-i)\ /L$
$\Delta Y = Y_D * \Delta L/L$ or $\Delta Y = Y_D * eh * \text{sine}(-i)\ /L$

$\Delta Z = Z_D * \Delta L/L$ or $\Delta X = X_D * eh * \text{cosine}(-i) / L$

These equations have a serious weakness, in that L can be zero (or close to it). This occurs at or near the North or South poles and needs to be considered during equation development. One such approach is presented in Appendix 1.

The coordinate shift from the X-Y-Z ECEF system to the AM (x-y-z) system) then consists of one movement from the Earth's center **O** to point **D**, plus another one from point **D** to point **C**, as follows:

$$X_C = X_D + \Delta X$$
$$Y_C = Y_D + \Delta Y$$
$$Z_C = Z_D + \Delta Z$$

Therefore, any point in the AM system has three sets of coordinates:

1. Within the geodetic system **ϕP**, **λP**, **eh**.
2. Within ECEF: **XP**, **YP**, **ZP**.
3. Within AM: **xP**, **yP**, **zP**.

Algorithm Development

Top goals for the development of the necessary relationships were:

1. Use the simplest-possible formulation.
2. Achieve the highest calculation speed.
3. Utilize existing and proven relationships to the maximum; and
4. Achieve the highest-possible precision (no series expansions, etc.).

The algorithm development starts with the given coordinates of AM point D, namely the geodetic coordinates (**φD**, **λD**, **ehD**) and its ECEF coordinates (**X**, **Y**, **Z**). The desired operation is to calculate the local AM coordinates of a point **P** (**xP**, **yP**, **zP**),

The development is done in three steps. First, the **AM** plane is established, then the position of point C is determined, followed by calculating the local coordinates for **P**.

Step 1 - Establishing the AM plane

See Figure 6a. A plane is considered that is vertical to the Equatorial plane that contains the ECEF **Z** axis, the line **L**, the geocenter **O** (as defined by ECEF), and point **D**. It also shows that vertical angle (i).

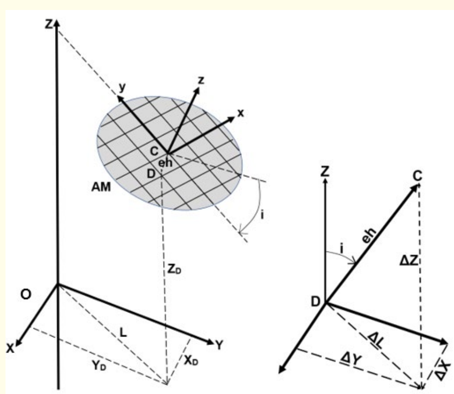

Figure 6-a ***Figure 6-b***

The following are all located in this vertical plane:

- ECEF Z-axis.
- Line L.
- ZD.
- eh.
- AM y axis.
- AM z axis.
- Angle i.

The inclination of the AM plane is the ellipse slope at D, or angle i.

At **C**, another plane is established (grey disk) that is parallel to the ellipsoid at **C** and perpendicular to the vertical plane. This is the **AM** plane. It has a Cartesian grid **x**, **y**, **z** so that the y axis always points towards the ECEF Z-axis (local Astronomic North). **C** is the origin of said grid and has local coordinates **x=0**, **y=0**, **z=0** (lower case).

One begins with calculating the ECEF X, Y, Z coordinates of **D** from geodetic coordinates, using the well-known relationships in Table 1. After that, one can formulate a three-dimensional Helmert (Orthogonal) transformation based on three shifts, three rotations and one scale factor.

Reference of existing algorithms:

$$e^2 = 2 * f - f^2$$

$$w = \sqrt{1 - e^2 * sin^2\,(\varphi)}$$

$$R_N = \frac{a}{w}$$

$$\boldsymbol{X} = (R_N + h)cos\,(\varphi)cos\,(\lambda)$$

$$\mathbf{Y} = (R_N + h)cos\,(\varphi)sin\,(\lambda)$$

$$\mathbf{Z} = \{(1-e^2)R_N + h\}sin(\varphi)$$

Table 1

Step 2 - Point C is calculated

The ECEF coordinates **X**, **Y**, **Z** of point **D** are calculated from the latitude, longitude and ellipsoidal height given for it. Points **C** and **D** share the same latitude and longitude, and differ by the ellipsoidal height. As a result, **D** is now known within a Cartesian coordinate system, which will allow the formulation of a three-dimensional Helmert (orthogonal) transformation, which can now be formulated. This is based on the three shifts, three rotations and one scale factor, as follows (see matrices on the next page).

- ***Shifts***: The three ECEF coordinates of Point **C** are the first three shifts from the ECEF origin to **D**. From the ECEF coordinates, the shifts (as shown in Figure 6a) are **XD**, **YD**, **ZD**. The second three shifts are from point **D** to point **C**, or **Δx**, **Δy**, **Δz**, as shown in Figure 6b. Note that the **Δz** shift can be positive or negative (above or below the ellipsoid, as defined by the ellipsoidal height **eh**).
- ***Rotations***: The three rotations between the ECEF system and the AM system are the following:
- The rotation **Ω** around the ECEF X axis is the angle **i**.
- The rotation **Φ** around the ECEF Y axis is always zero since the x axis of **AM** always remains parallel to the Equatorial plane (perpendicular to the ECEF Z axis).

- The rotation **K** around the ECEF Z axis is the longitude of point **D**. Western longitudes are entered as negative values.

These rotations are placed into a 3x3 rotation matrix **M** as shown in Figure M1 below, with these **Ω**, **Φ**, **K** rotations. Figure M2 shows the rotations replaced by the angles **i** and **λD** (the longitude of point **D**). Note that this is an “orthogonal matrix” (the inverse is equal to its transpose, which is especially practical for geodetic inverse calculations).

$$\begin{vmatrix} \cos\Omega\cos K & \sin\Omega\sin\Psi\cos K + \cos\Omega\sin K & -\cos\Omega\sin\Psi\cos K + \sin\Omega\sin K \\ -\cos\Psi\sin K & -\sin\Omega\sin\Psi\sin K + \cos\Omega\cos K & \cos\Omega\sin\Psi\sin K + \sin\Omega\cos K \\ \sin\Psi & -\sin\Omega\cos\Psi & \cos\Omega\cos\Psi \end{vmatrix}$$

Figure M1

(or)

$$\begin{vmatrix} \cos \mathbf{i} \cos \boldsymbol{\lambda}_{\mathbf{D}} & \cos \mathbf{i} \sin \boldsymbol{\lambda}_{\mathbf{D}} & \sin \mathbf{i} \sin \boldsymbol{\lambda}_{\mathbf{D}} \\ -\sin \boldsymbol{\lambda}_{\mathbf{D}} & \cos \mathbf{i} \cos \boldsymbol{\lambda}_{\mathbf{D}} & \sin \mathbf{i} \cos \boldsymbol{\lambda}_{\mathbf{D}} \\ 0 & -\sin \mathbf{i} & \cos \mathbf{i} \end{vmatrix}$$

Figure M2

- ***Scale factor***: Since the transformation is from one metric Cartesian system to another (ECEF to AM), the scale factor has a value of exactly one (1).

Step 3 - The position of a point P is calculated.

Consider a point **P** that is near to point **C** (as shown in Figure 7). This point **P** has its own latitude, longitude, and ellipsoidal height. Note that the term “near” is relative, in that C is within the intended boundaries of the AM plane.

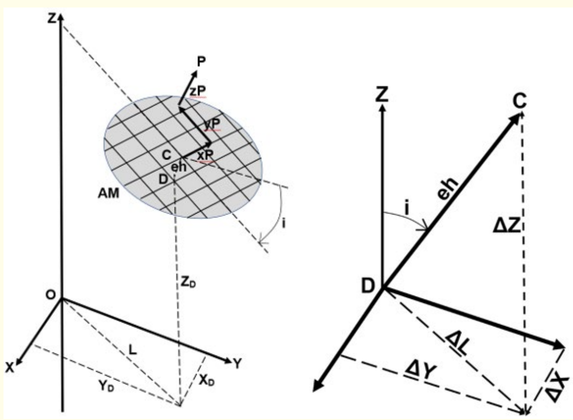

Figure 7-a ***Figure 7-b***

To get from origin **O** to **P**. one must first shift from **O** to **D** (X_D, Y_D, Z_D), then from **D** to **C** (**ΔX**, **ΔY**, **ΔZ**), and then through the rotated **xP**, **yP**, **zP**.

The transformation converts the **AM** coordinates of point **P** to ECEF coordinates. The general transformation formulation is the following:

$$\begin{vmatrix} \mathbf{XP} \\ \mathbf{YP} \\ \mathbf{ZP} \end{vmatrix} = \begin{vmatrix} \mathbf{XC} \\ \mathbf{YC} \\ \mathbf{ZC} \end{vmatrix} + \mathrm{M} \begin{vmatrix} \mathbf{xP} \\ \mathbf{yP} \\ \mathbf{zP} \end{vmatrix}$$

Where the rotation matrix M, as shown in Figure M2, is applied to the AM coordinates **xP**, **yP**, **zP**, and the shift from point **O** to point **C** is given as **XC**, **YC**, **ZC**.

Then an inverse transformation is applied

If one wants to convert ECEF coordinates to AM ones, one reverses that equation to this:

$$\begin{vmatrix} \mathbf{xP} \\ \mathbf{yP} \\ \mathbf{zP} \end{vmatrix} = \mathbf{M^T} \begin{vmatrix} \mathbf{XP - XC} \\ \mathbf{YP - YC} \\ \mathbf{ZP - ZC} \end{vmatrix}$$

Note that the transpose of the M matrix is used (since for orthogonal matrices, the transpose is equal to the inverse), instead of the rotation angles being changed in sign. This allows the inverse calculation to be done without having to worry about the signs of the original latitude and longitude of point C.

Notes

1. Given that this solution is intended only for relatively small areas (like a small county, a municipality, a large project, a space station, etc.), a few simple shortcuts were made. The error level is estimated to be in the centimeter range. See note below about radii of curvature.
2. Refinements to this solution will be identified and introduced as they become available.
3. An inverse solution is provided herein, to make this a tool that is applicable in the real world. Therefore, the following was considered:
   - Within any given small area, this **AM** reference is established once with the help of GPS observations. This results in (initially) geodetic coordinates for the center point **C**, and the placement of the **AM** plane. Also, astronomic North will have to be established for point **C**.
   - Any other points **P** (such as township or section corners, measured or looked up) are placed into the **AM** georeference.
   - Any placement within the **AM** of project data is done using Total Stations, etc., in meters, without any scale change. If measurements in feet (or US Survey Feet) are needed, this can be scaled directly from the center point **C**.
   - In other words, this solution would provide Engineers, Surveyors, and GIS practitioners a common reference frame that works at any average elevation and that does not require any projections and/or scale factors.
4. All current and existing geodetic constructs remain in place:
   - All local, regional, national, or international datums.
   - All local and international projections.
   - All current ECEF definitions and implementations.
   - There is no need to redefine or change anything.
5. Regarding elevations:
   - All projections are two-dimensional solutions, in that they result in x, y (or N, E) positions. No projections provide "z" or elevation coordinates. In contrast to that, AM provides x, y, z positions, where the "z" coordinates are heights above or below the **AM** plane. One elevation (orthometric) controls everything, and that is the one for point **C**.
6. Regarding radii of curvature, there are various types, such as the meridional radius of curvature (M), the normal radius of curvature (N), a mean radius of curvature, etc., and all of them depend on the ellipsoid used and the latitude of a location, as follows:

- *Meridional radius of curvature*. This is the radius of curvature of the ellipsoidal meridian at a given point. This radius changes continuously, since it has a minimum value at the Equator, and a maximum value at the poles. Literature identifies this radius as “RM” or “M”.
- *Normal radius of curvature*. This is also a radius of curvature through a point on the ellipsoid but extending East-West (perpendicular or “normal”) to the local meridian. Its size is opposite to the meridional radius, since its maximum value is at the poles, and the minimum is at the Equator (the Equator itself is an arc normal to meridians). At the poles, meridional and normal radii are the same. Literature identifies this radius as “RN” or “N”.
- *Other radii*. Radii that are not meridional or normal vary in between. The biggest combined change is realized mid-latitude.
- For this AM solution, none of them are used for AM. Only the normal radius (RN) appears in geodetic calculations, for two reasons:
- The accepted calculation of ECEF coordinates is based on RN only, and
- The differences between RM and RN are significant, but for the small areas covered by AM they are negligible. For example, for a latitude of 39 degrees, they differ by 0.4 %. As such, the AM equations follow the standard set by ECEF.

7. The **AM** workspace can be used at any point in relation to the ellipsoid. There are no limitations in relation to latitude or longitude. It can be used near the North or South Poles if the formulation respects the comments made in Appendix 1 regarding the calculation of the angle “**i**”.

Conclusions

The AltiMetric System is introduced as a working space for locales that are vertically far removed from normal geodetic constructs such as projections. While it uses a mix of existing geometric, trigonometric, and geodetic constructs, AM creates a three-dimensional Cartesian workspace that can be positioned to fit an average ground elevation. This means that Engineers, Surveyors, and GIS Operators can work within the same common framework, without having to resort to projections, scale factors, modified coordinates, etc. This is of special interest to people living at higher elevations, such as Denver.

If one wishes to establish a similar solution based on feet, and not meters, then one simply replaces the above scale factor of 1 with 0.3048 or similar.

At the same time, the AM solution possibly represents the easiest and simplest form of a workspace that competes favorably with projections. AM coordinates can easily be converted to other quantities, such as conventional projection or ECEF data using existing math and software.

Notes about elevation

Existing projections, as they are implemented for regional horizontal datums such as any of the NAD versions, result in the creation of a local grid (local within a zone), but the “Z” value fits within a different geo-reference. These orthometric elevations are referred to and perpendicular to a specific geoid. As such, elevations are not perpendicular to a state plane X, Y grid, and therefore these X, Y, Z georeferences do not form a Cartesian system, even if often they are treated as such.

As mentioned above, AM is a Cartesian system, and therefore its “z” values do not correspond to elevations. How can one convert these “z” values to elevations?

The reader is reminded that AM is a plane, and not a curved surface in 3D space that is developed into a plane surface, as with projections. As such, and as a minimum, any earth curvature correction needs to be part of any conversion between AM “z” coordinates and real elevations. In addition, undulations of the geoid should also be respected, and the corresponding geoidal height corrections need to be applied.

However, that said, if the Altimetric Workspace only covers an area of relatively small extent, Then the deflections of the vertical can be considered negligible. In such a situation, the Workspace can be accompanied by existing elevation data such as the Digital Elevation Model (DEM) as provided by the USGS.

Claim

A copyright was received for this work. See Appendix 2. The following is claimed:

- An algorithm to establish a local coordinate system for high-elevation locations, comprising of:
- Inserting a new algorithm into the conventional geodetic workflow that creates a local Cartesian coordinate system.
- Defining the location of the origin (0,0,0) of said local coordinate system based on a given latitude, longitude and ellipsoidal height.
- Calculating the position of a geometric plane, based on the latitude, longitude, and ellipsoidal height of said origin, so that the plane is parallel to the ellipsoid at that origin, and so that mappers can use that plane as a local, three-dimensional, Cartesian coordinate system.
- Given that said origin can be established at any ellipsoidal height, establishing a plane at any elevation, above or below ground, or in outer space, on a different celestial body, as long as an ECEF coordinates system is nearby.
- Establishing a coordinate system (x, y, z) so that the specified point is its origin and a North axis point toward the rotation axis of the ECEF system as defined by a given datum ellipsoid.
- Calculating the local coordinates of any other point from its latitude, longitude and ellipsoidal height, resulting in a local set of coordinates of x, y, and z.
- Generally, significantly simplifying the creation of a local Cartesian georeference for geographic information systems and building a surface that can be used directly by engineers and surveyors.

Final Comments

This paper presents a workspace that allows the capture of three-dimensional data. This idea alone is not new, since Google Earth offers “Street View”, a tool that is used to display three dimensional views of its database.

However, Google Earth has so far not disclosed the accuracy of its data. The best estimates available so far by others peg its positional accuracy to about ±2 feet. This does not allow the capture and display of accurate survey and/or engineering data.

This shortcoming justified the development of the alternative presented herein.

The use of matrices herein indicates that these solutions are best implemented with the help of matrix calculus. However, they are simple enough to be calculated with a hand-held calculator.

Data Availability

All data, models, and code generated or used during the study appear in this article as submitted.

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Appendix 1 - Altimetric Plane Inclination

In terms of rotations of the AM plane, the following are the two basic facts:

- The horizontal orientation of the AM plane is so that its y-axis always points exactly to earth's axis of rotation, and
- The vertical tilt of the AM plane has an angle "**i**" with respect to the Equatorial Plane. See Figure A1. Or the z-axis of AM makes an angle i with the geocentric Z-axis.

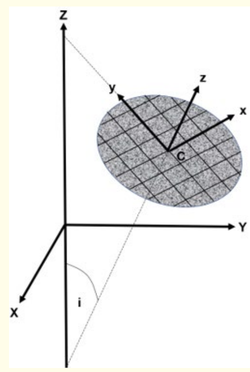
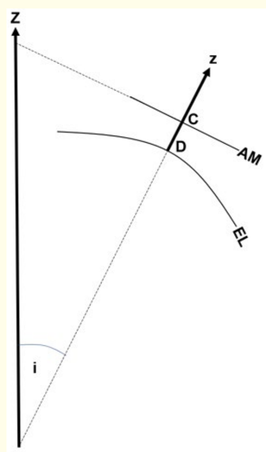


Figure A1

The next step is to determine this AM plane slope i.

At point **C**, the AM plane is parallel to the Earth's ellipsoid. If one looks along the AM x-axis, one gets the view as presented in Figure A2 and A3. The ellipse represents the meridian running through point **C**.



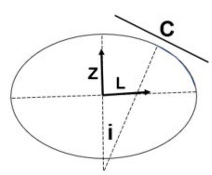


Figure A2 ***Figure A3***

To calculate angle i, one uses the Standard Formula for an ellipse, or:

$$\frac{L^2}{a^2} + \frac{Z^2}{b^2} = 1$$

These L, Z values correspond to the position of point C on the ellipse, where Z is the geocentric coordinate of C. L is the Pithagorean combination of the geocentric coordinates X_C and Y_C. Thus: $L = \pm\sqrt{X_C{}^2 + Y_C{}^2}$

The equation of the ellipse is differentiated with respect to L and set to zero. The resulting differential will be the AM slope S.

$$\frac{df}{dL} = \frac{2L}{a^2} + \frac{2Z}{b^2}\frac{dZ}{dL} = 0$$

$$\frac{Z}{b^2}\,\frac{dZ}{dL} = -\frac{L}{a^2}$$

$$\frac{dZ}{dL} = -\frac{Lb^2}{Za^2} = \text{Slope S}$$

It has to be noted that Z can be zero, which would make the slope infinite. This would be the slope at the Equator. It is suggested that at or near the Equator, the reciprocal slope 1/i should be used (similar to tangent/cotangent treatment).

In this case we will use the semi-major and semi-minor axes of GRS80, or (a, b) = (6,378,137.000m, 6,356,752.3m). These are inserted into the above equation.

The above equation has one critical condition, and that is when Z = 0, or close to. This means that a workaround must be devised. One solution is to sometimes change the equation to the following:

$$\frac{dZ}{dL} = -\frac{Za^2}{Lb^2} = \text{Complement of slope S}$$

The conversion of the slope to the necessary rotation angle "I" would be one of the following, where one chooses the first for locations of point C where | L | <= | Z |, and the other where | L | > | Z | (boundary arbitrarily chosen, see shaded areas in Figure A4). This allows a seamless solution range from 0° - 180°. The corresponding equations are:

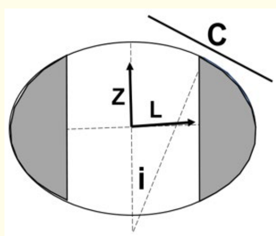


Figure A3

When | L | <= | Z |: $i = \arctan\left(-\frac{Za^2}{Lb^2}\right)$ or

When | L | > | Z |: $i = 90 - \arctan\left(-\frac{Lb^2}{Za^2}\right)$.

Appendix 2 - A sample calculation

The purpose of his appendix is to illustrate a sample calculation of the direct and inverse conversion of one point P. The results are presented in the Microsoft Excel Table T2 that is found below.

In this example, the general approach is to:

1. Select an ellipsoid.
2. Select a set of geodetic coordinates for the AM center **C**.
3. Select a seed value for the distance of a point **P** from **C** (in 3d) and calculating the geodetic coordinates of **P** based on that.
4. Use these geodetic cords for C and P to calculate sample AM coordinates (x, y, z).

This table contains the following:

- Two ellipsoidal parameters **a** and **f**, the second eccentricity e2, the value pi, and one seed value for the calculations (approximating 1000m on the ellipsoid).
- Input fields for latitude, longitude and ellipsoidal height (Lat, Long, eh) for point C (the AM origin), and the calculated ECEF coordinates for point C.
- Calculated values for the two radii of curvature MC and NC (or RM and RN at point C); followed by calculated values for the differences in latitude and longitude between points C and P.
- The latitude phiP, longitude lambP for point P (calculated from the seed values), ellipsoidal height of P (a value that is entered), the intermediate value w; and the calculated ECEF coordinates of point P (XP, YP, ZP).
- Below that, are the direct and inverse calculations using:
- The rotation values **Ω**, **Φ**, **κ** for the AM plane (calculated from the phiC, lamC, and ehC values above).
- The rotation matrix based on the **Ω**, **Φ**, **κ** values in the grey fields.
- The difference in ECEF coordinates between C and P (dX, dY, dZ).
- The three shift values T1, T2, T3 (as in section 1.2).
- The resulting AM coordinates for P.
- The inverse calculations use the shift values that are the same as the C coordinates.

In both direct and inverse calculations checks are performed. As can be seen, the differences between starting coordinates of point P are the same as the inverse results.

ALTIMETRIC CONVERSION

a	f	e2	π	SEED [m]			
6378137.000	0.00335	0.00669	3.141593	999.217			
phiC	lamC	ehC	wC	RNC	XC	YC	ZC
39.0000000	-105.0000000	5000.000	0.998673	6386608.932	-1285609.343	-4797959.387	3995463.625
MC	NC	Δφ	Δλ				
6360718.527	6386608.932	0.0090007	0.0115348				
phiP	lamP	ehP	wP	RNP	XP	YP	ZP
39.0090007	-104.9884652	10000.000	0.998673	6386612.225	-1285485.436	-4801363.190	3999387.936
						P-C global >	5196.291
DIRECT							
Ω	φ	K			dX	dY	dZ
-51.0000000	0.0000000	15.0000000			123.907	-3403.803	3924.311
m11	m12	m13 >	0.966	-0.259	0.000	T1 >	0.000
m21	m22	m23 >	0.163	0.608	0.777	T2 >	0.000
m31	m32	m33 >	-0.201	-0.751	0.629	T3 >	0.000
					xP	yP	zP
INVERSE					1000.654	1000.852	4999.843
Ω	φ	K				P-C local >	5196.291
-51.0000000	0.0000000	15.0000000					
m11	m12	m13 >	0.966	0.163	-0.201	T1 >	-1285609.343
m21	m22	m23 >	-0.259	0.608	-0.751	T2 >	-4797959.387
m31	m32	m33 >	0.000	0.777	0.629	T3 >	3995463.625
					XP	YP	ZP
					-1285485.436	-4801363.190	3999387.936
				diffs->	0.000	0.000	0.000
						P-C global >	5196.291

Table T2