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Star-Laplace Transform S-Step

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Abstract

The Laplace transform has many applications in science and engineering because it is a tool for solving differential equations; in this paper we propose a Star-Laplace transform s-step. We give the definition of this Star-Laplace transform *s*-step of function $f(t)$, $t \in [0,+\infty]$, some examples and basic properties. We also give the form of its inverse by using the theory of the Laplace Transform.

Keywords and phrases: Star Laplace Transform s-step; Laplace Transform; Star Coefficient; Star-System; Equations Matrix

Introduction

The Laplace transform is used frequently in engineering and physics; The Laplace transform can also be used to solve differential equations and is used extensively in mechanical engineering and electrical engineering. The Laplace transform reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra. We consider algebraic properties as well as more abstract properties such as realizing that the Star-System with α Coefficient (see [1, 2 and 3]) has a unique solution is called the star-solution or star-function. When we solve a linear Star-System with α Coefficient in five unknowns $\star[a;b;c;d;e;\alpha] = \alpha$, we will place the 5 Star-Vectors that are the solution (linearly independent) in the columns of a matrix, So what we've done is create the Star-Coefficient $\alpha^\star$.

In this paper, we define (SLT) the Star-Laplace transform s-step of fairly regular function $f : \mathbb{R}_+ \to \mathbb{C}$ by:

$$\mathcal{L}_s^\star f(\omega) = \int_0^{+\infty} (\alpha_s^\star f)(t) e^{-i\omega t} dt. \tag{1.1}$$

We illustrate this criterion of (definition SLT) by providing some applications.

The contents of the paper are as follows:

In the second section, we present some preliminary results and notations that will be useful in the sequel ([1 and 2]).
In the last section, we defined the Star Laplace Transform s-step (SLT) and certain of its Applications.

Some basic definitions and notations

Laplace Transform

Definition 1. Let $\omega = a + ib \in \mathbb{C}$ and $f : \mathbb{R}_+ \to \mathbb{C}$ measurable such that the function $t \mapsto f(t)\, e^{-\omega t}$ belongs to soit dans $L^1(\mathbb{R}_+, dt)$. The Laplace transform of f noted $\boldsymbol{L}(f)$ is given by:

$$\mathcal{L}(f)(\omega) = \int_0^{+\infty} f(t)\, e^{-\omega t} dt. \tag{2.1}$$

The function $\omega \mapsto \omega L(f)(\omega)$ is called Carson transform of *f. Th* map $f \mapsto L(f)$ is named the Laplace transformation . If $F = L(f)$, we write.

$$\forall t \geq 0, f(t) = L^{-1}(F)(t).$$

One can prove that the Laplace transform L is injective (see page 9 in [8]), that is the reason why L^{-1} is well defined (for a precise formula of L^{-1}, see page 10 in [8]).

To compute Laplace transforms, we need:

$$d(fg) = fdg + gdf, \quad \int_a^b df = f(b) - f(a), \tag{2.2}$$

Where $df := f'(t)dt$.

Laplace transform is linear: By linearity, we mean for all real numbers a, b,

$$L(af + bg) = aL(f) + bL(g)$$

Following pages, we introduce some notations and star-system with α coefficient defined in [1] and [2].

A star-system with α coefficient

Definition 2. Let a, b, c, d, e and α be real numbers, and let T_1, T_2, T_3, T_4, T_5 be unknowns (also called variables or indeterminates). Then a system of the form

$$\begin{cases} T_1 + T_2 = \alpha - a - c \\ T_2 + T_3 = \alpha - b - d \\ T_3 + T_4 = \alpha - c - e \\ T_4 + T_5 = \alpha - a - d \\ T_1 + T_5 = \alpha - b - e \end{cases}$$

The scalars a, b, c, d, e are called the coefficients of the unknowns, and α is called the constant "*Chaf f*" of the star-system in five unknowns. A vector $(T_1, T_2, T_3, T_4, T_5)$ in R^5 is called a star-solution vector of this star-system if and only if $\star[a,b,c,d,e;\alpha] = \alpha$.

The solution of a Star-system is the set of values for T_1, T_2, T_3, T_4 and T_5 that satisfies five equations simultaneously.

A star-element

A star-element (see: [1, 2, 4, 5, 6, 7]) is a term of the five-tuple $(T_1, T_2, T_3, T_4, T_5)$ solution of a star-system $\star[a,b,c,d,e;\alpha] = \alpha$, wher $(T_1, T_2, T_3, T_4, T_5) \in \mathbb{R}^5$.

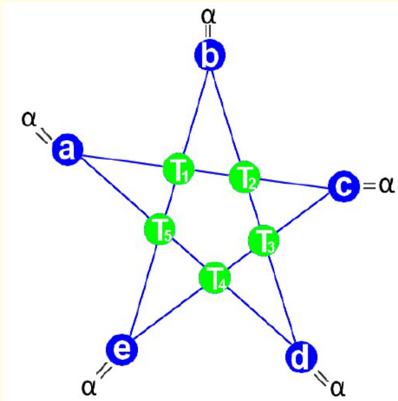

Figure 1: In addition to having the sum α in each line.
Is called a star-system with coefficient α in five unknowns.
We have also noted $\star[a, b, c, d, e; \alpha] = \alpha$.

Star-Coefficient

The star-Coefficient (see [1]) is also noted by $\alpha_\star$ and is a solution of equation

$$\alpha = T_1(\alpha) + T_2(\alpha) + T_3(\alpha) + T_4(\alpha) + T_5(\alpha),$$

Where $(T_1, T_2, T_3, T_4, T_5)$ is solution of a star-system:

$$\star[a, b, c, d, e; \alpha] = \alpha$$

On the other hand, the star-system with coefficient α (see [1]) can be written in matrix form $T^\star = M^\star F^\star$ where

$$M^\star = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{4}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{4}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} & \frac{4}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}, \ T^\star = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} \text{ and } F^\star = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix}.$$

$M^\star$ is called the star-Matrix of the star-system with α coefficient

$$\star[f_1, f_2, f_3, f_4, f_5; \alpha] = \alpha.$$

Set-Star

The set-star is constructed from the solution set of linear star-system with coefficient α ($\star[a,b,c,d,e;\alpha] = \alpha$). The Set-star will be noted by $S^\star$.

Star-System equivalent

Equivalent Star-Systems (see [1]) are those systems having exactly same solution, i.e. two star-systems are equivalent if solution of one star-system is the solution of other, and vice-versa.

Star-Laplace Transform s-step

Consider the following star-system:

$$\star[f(t+s), f(t+2s), f(t+3s), f(t+4s), f(t+5s); \alpha] = \alpha$$

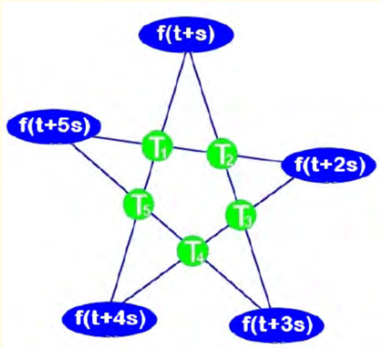

Figure 2: In addition to having the sum $\alpha^{\star}_{t,s}$ in each line.

of five equations in five unknowns:

$$\begin{cases} T_1 + T_2 = \alpha - f(t+5s) - f(t+2s) \\ T_2 + T_3 = \alpha - f(t+s) - f(t+3s) \\ T_3 + T_4 = \alpha - f(t+2s) - f(t+4s) \\ T_4 + T_5 = \alpha - f(t+3s) - f(t+5s) \\ T_5 + T_1 = \alpha - f(t+4s) - f(t+s)) \end{cases}$$

In a particular case if $\alpha = T_2(\alpha) + T_2(\alpha) + T_3(\alpha) + T_4(\alpha) + T_5(\alpha)$, we obtain:

$$T^{\star} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{4}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{4}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} & \frac{4}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} f(t+s) \\ f(t+2s) \\ f(t+3s) \\ f(t+4s) \\ f(t+5s) \end{pmatrix}.$$

In this case, we have

$$\alpha^{\star}_{s} f(t) = \frac{2}{3}\left(f(t+s) + f(t+2s) + f(t+3s) + f(t+4s) + f(t+5s)\right).$$

Behold, the Star-Laplace Transform s-step is born!

To productively use the Star-Laplace Transform, we need to be able to transform functions from the time domain to the Star-Laplace domain.

Definition 3. We say that a function $f : \mathbb{R} \to \mathbb{C}$ is causal if for all $t < 0$, we have $f(t) = 0$.

Remark 1. The Heaviside function (or the unit step) defined as

$$H(t)=\begin{cases}0, & if \quad t<0\\ 1, & if \quad t\geq 0\end{cases} \tag{3.1}$$

Allows to “manufacture” causal functions.

Definition 4. Let ω = *a* + *ib* ∈ ℂ, *s* ∈ ℝ and *f* : ℝ→ ℂ be causal and fairly function. The Star-Laplace transform s-step (SLT) of *f* noted is given by:

$$\mathcal{L}_s^\star f(\omega)=\int_0^{+\infty}(\alpha_s^\star f)(t)e^{-\omega t}dt. \tag{3.2}$$

Let’s take a specific, simple, and important example.

Example 1. The Unit Step Function

Let *H*, the unit step function given by the relation (3.1). The unit step function is defined as

$$H(t)=\begin{cases}0 \; if \; t<0\\ 1 \; if \; t\geq 0\end{cases}$$

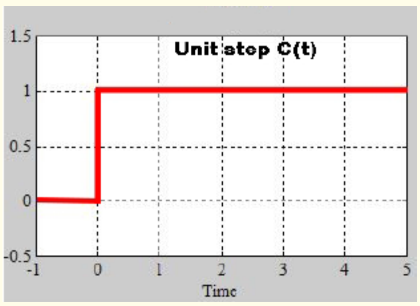

Figure 3: The graph of the unit step function *H* (*t*).

To find the Star-Laplace Transform s-step, we apply the definition (3.2). For all ω >0, we have

$$\begin{aligned}
\mathcal{L}_s^\star H(\omega) &= \int_0^{+\infty}(\alpha_s^\star H)(t)e^{-\omega t}dt\\
&= \frac{2}{3}\int_0^{+\infty}\left[H(t+s)+H(t+2s)+H(t+3s)+H(t+4s)+H(t+5s)\right]e^{-\omega t}dt\\
&= \frac{2}{3}\int_s^{+\infty}H(x)e^{-\omega(x-s)}dx+\frac{2}{3}\int_{2s}^{+\infty}H(x)e^{-\omega(x-2s)}dx+\frac{2}{3}\int_{3s}^{+\infty}H(x)e^{-\omega(x-3s)}dx\\
&+ \frac{2}{3}\int_{4s}^{+\infty}H(x)e^{-\omega(x-4s)}dx+\frac{2}{3}\int_{5s}^{+\infty}H(x)e^{-\omega(x-5s)}dx\\
&= \frac{2}{3}\left(e^{\omega s}G(s)+e^{2\omega s}G(2s)+e^{3\omega s}G(3s)+e^{4\omega s}G(4s)+e^{5\omega s}G(5s)\right)
\end{aligned}$$

Where

$$\forall y \in \mathbb{R},\ \ G\left(y\right) = \int_{y}^{+\infty} H(x)e^{-\omega x}dx.$$

We have

$$\forall y \in \mathbb{R},\ \ G\left(y\right) = \begin{cases} \frac{1}{\omega}, & if \quad y < 0 \\ \frac{e^{-\omega y}}{\omega}, & if \quad y \geq 0 \end{cases}$$

Then for all s ∈ ℝ, we obtain

$$\forall \omega > 0, \mathcal{L}_{s}^{\star} H(\omega) = \begin{cases} \frac{2}{3\omega}\left(e^{\omega s} + e^{2\omega s} + e^{3\omega s} + e^{4\omega s} + e^{5\omega s}\right), & if \quad s < 0 \\ \frac{10}{3\omega}, & if \quad s \geq 0. \end{cases}$$

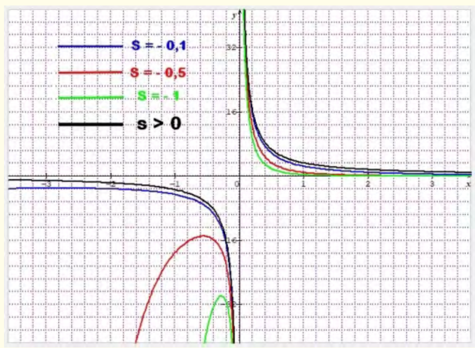

Figure 4: The Star-Laplace Transform s-step of the unit step function, if (s= -0, 1), if (s = −0, 5), if (s = −1) and if s > 0.

Example 2. The Sine Function

Now, we consider the caual function f_{ξ}, $\xi \in \mathbb{R}*$ given by:

$$\forall t \in \mathbb{R},\ \ f_{\xi}\left(t\right) = \sin(\xi t)H(t) = \begin{cases} 0, & if \quad t < 0 \\ \sin(\xi t), & if \quad t \geq 0. \end{cases}$$

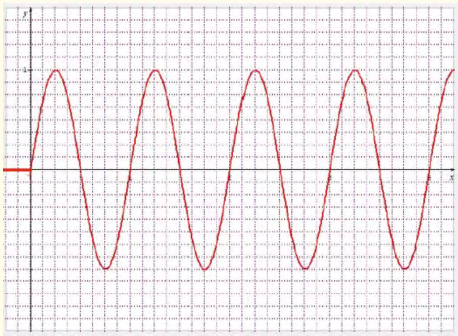

Figure 5: The graph of the $f_{2\pi}(t)$.

As before, start with the definition of the Star-Laplace Transform s-step for all ω >0, we have

$$\begin{aligned} &= \frac{2}{3}\int_0^{+\infty}\{H(t+s)\sin(\xi(t+s))+H(t+2s)\sin(\xi(t+2s))+H(t+3s)\sin(\xi(t+3s)) \\ &+H(t+4s)\sin(\xi(t+4s))+H(t+5s)\sin(\xi(t+5s))\}e^{-\omega t}dt \\ &= \frac{2}{3}\int_s^{+\infty}H(x)\sin(\xi x)e^{-\omega(x-s)}dx+\frac{2}{3}\int_{2s}^{+\infty}H(x)\sin(\xi x)e^{-\omega(x-2s)}dx \\ &+\frac{2}{3}\int_{3s}^{+\infty}H(x)\sin(\xi x)e^{-\omega(x-3s)}dx+\frac{2}{3}\int_{4s}^{+\infty}H(x)\sin(\xi x)e^{-\omega(x-4s)}dx \\ &+\frac{2}{3}\int_{5s}^{+\infty}H(x)\sin(\xi x)e^{-\omega(x-5s)}dx \\ &= \frac{2}{3}\left(e^{\omega s}G_\xi(s)+e^{2\omega s}G_\xi(2s)+e^{3\omega s}G_\xi(3s)+e^{4\omega s}G_\xi(4s)+e^{5\omega s}G_\xi(5s)\right) \end{aligned}$$

Where

$$\forall y\in\mathbb{R},\ G_\xi(y)=\int_y^{+\infty}H(x)\sin(\xi x)e^{-\omega x}dx.$$

Since

$$\forall\omega>0,\ \forall z\in\mathbb{R},\ \int_0^z\sin(\xi x)e^{-\omega x}dx=\frac{\xi}{\omega^2+\xi^2}-\frac{e^{-\omega z}}{\omega^2+\xi^2}\left[\xi\cos(\xi z)+\omega\sin(\xi z)\right]$$

We obtain

$$\forall y\in\mathbb{R},\ G_\xi(y)=\begin{cases}\frac{\xi}{\omega^2+\xi^2}, & if\quad y<0\\ \frac{e^{-\omega y}}{\omega^2+\xi^2}\left[\xi\cos(\xi y)+\omega\sin(\xi y)\right], & if\quad y\geq 0\end{cases}.$$

Then for all $s\in\mathbb{R}$ and $\omega>0$, we get

$$\mathcal{L}_s^\star(f_\xi)(\omega)=\begin{cases}\frac{2\xi}{3(\omega^2+\xi^2)}\left(e^{\omega s}+e^{2\omega s}+e^{3\omega s}+e^{4\omega s}+e^{5\omega s}\right), & if\quad s<0\\ \frac{2\xi}{3(\omega^2+\xi^2)}\left[\cos(s\xi)+\cos(2s\xi)+\cos(3s\xi)+\cos(4s\xi)+\cos(5s\xi)\right] & \\ +\frac{2\omega}{3(\omega^2+\xi^2)}\left[\sin(s\xi)+\sin(2s\xi)+\sin(3s\xi)+\sin(4s\xi)+\sin(5s\xi)\right], & if\quad s\geq 0\end{cases}$$

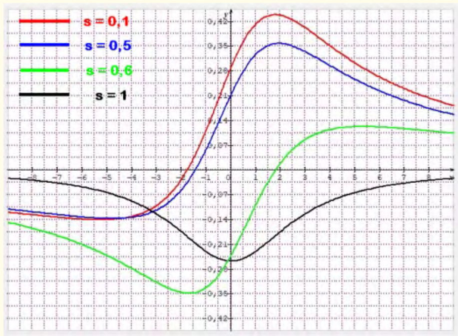

Figure 6: The graph of the Star-Laplace Transform s-step $L^\star_s f\pi(t)$.

If s = 1 and ξ = π, we get

$$\mathcal{L}_1^\star f_\pi(\omega) = \frac{2\pi}{3(\omega^2+\pi^2)} = \det(M^*)\mathcal{L}(\sin(\pi t))(\omega).$$

If s = 1 and ξ = $\frac{\pi}{2}$, we obtain

$$\mathcal{L}_1^\star f_{\frac{\pi}{2}}(\omega) = \frac{8\omega}{3(4\omega^2+\pi^2)} = \det(M^*)\mathcal{L}(\cos(\frac{\pi}{2}t))(\omega).$$

In general, we can deduce from the simple observation the following results

(1) *If* $s = 1$ *and* $\xi = (2n+1)\pi,\ n \in \mathbb{R}$

$$\mathcal{L}_1^\star f_{(2n+1)\pi}(\omega) = \det(M^*)\mathcal{L}(\sin(\pi t))(\omega).$$

(2) *If* $s = 1$ *and* $\xi = \frac{\pi}{2} + 2n\pi,\ n \in \mathbb{R}$

$$\mathcal{L}_1^\star f_{\frac{\pi}{2}+2n\pi}(\omega) = \det(M^*)\mathcal{L}(\cos(\frac{\pi}{2}t)(\omega))$$

Example 3. The hyperbolic sine function

Let ξ > 0 and g_ξ the caual function given by:

$$\forall y \in \mathbb{R},\ g_\xi(t) = sh(\xi t)H(t) = \begin{cases} 0, & if \quad t < 0 \\ \frac{1}{2}\left(e^{\xi t} - e^{-\xi t}\right), & if \quad t \geq 0. \end{cases}$$

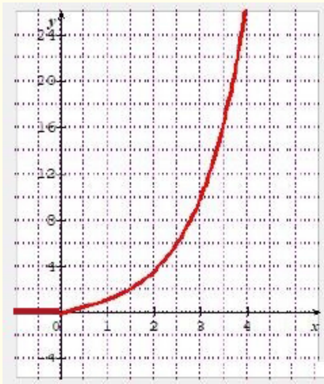

Figure 7: The graph of the $g_1(t)$ function.

For all ω > ξ, we have

$$\begin{aligned}
\mathcal{L}_s^{\star}\left(g_{\xi}\right)(\omega) &= \int_0^{+\infty} (\alpha_s^{\star} f_{\xi})(t) e^{-\omega t} dt \\
&= \frac{2}{3}\int_0^{+\infty} \left\{ H(t+s) sh\left(\xi(t+s)\right) + H(t+2s) sh\left(\xi(t+2s)\right) + H(t+3s) sh\left(\xi(t+3s)\right) \right. \\
&\quad \left. + H(t+4s) sh\left(\xi(t+4s)\right) + H(t+5s) sh\left(\xi(t+5s)\right) \right\} e^{-\omega t} dt \\
&= \frac{2}{3}\int_s^{+\infty} H(x) sh\left(\xi x\right) e^{-\omega(x-s)} dx + \frac{2}{3}\int_{2s}^{+\infty} H(x) sh\left(\xi x\right) e^{-\omega(x-2s)} dx \\
&\quad + \frac{2}{3}\int_{3s}^{+\infty} H(x) sh\left(\xi x\right) e^{-\omega(x-3s)} dx + \frac{2}{3}\int_{4s}^{+\infty} H(x) sh\left(\xi x\right) e^{-\omega(x-4s)} dx \\
&\quad + \frac{2}{3}\int_{5s}^{+\infty} H(x) sh\left(\xi x\right) e^{-\omega(x-5s)} dx \\
&= \frac{2}{3}\left(e^{\omega s} K_{\xi}(s) + e^{2\omega s} K_{\xi}(2s) + e^{3\omega s} K_{\xi}(3s) + e^{4\omega s} K_{\xi}(4s) + e^{5\omega s} K_{\xi}(5s) \right)
\end{aligned}$$

Where

$$\forall y \in \mathbb{R},\ \ K_{\xi}(y) = \int_y^{+\infty} H(x) sh\left(\xi x\right) e^{-\omega x} dx.$$

Since

$$\forall \omega > 0,\ \forall z \in \mathbb{R},\ \int_0^z sh\left(\xi x\right) e^{-\omega x} dx = \frac{1}{2}\left(\frac{e^{(\xi-\omega)z}}{\xi-\omega} + \frac{e^{-(\xi+\omega)z}}{\xi+\omega} \right) + \frac{\xi}{\omega^2-\xi^2}.$$

We obtain

$$\forall y \in \mathbb{R},\ \ K_{\xi}(y) = \begin{cases} \frac{\xi}{\omega^2-\xi^2}, & if \quad y < 0 \\ -\frac{1}{2}\left(\frac{e^{(\xi-\omega)y}}{\xi-\omega} + \frac{e^{-(\xi+\omega)y}}{\xi+\omega} \right), & if \quad y \geq 0 \end{cases}.$$

Then for all $s \in \mathbb{R}$ and $\omega > 0$, we get

$$\mathcal{L}_s^{\star}(f_\xi)(\omega) = \begin{cases} \frac{2\xi}{3(\omega^2-\xi^2)}\left(e^{\omega s} + e^{2\omega s} + e^{3\omega s} + e^{4\omega s} + e^{5\omega s}\right), & if \quad s < 0 \\ \frac{2\xi}{3(\omega^2-\xi^2)}\left[ch(s\xi) + ch(2s\xi) + ch(3s\xi) + ch(4s\xi) + ch(5s\xi)\right] \\ +\frac{2\omega}{3(\omega^2-\xi^2)}\left[sh(s\xi) + sh(2s\xi) + sh(3s\xi) + sh(4s\xi) + sh(5s\xi)\right], & if \quad s \geq 0 \end{cases}$$

Where

$$\forall t \in \mathbb{R}, \ ch(t) = \frac{1}{2}\left(e^{t} + e^{-t}\right).$$

This results show a relationship between the Star-Laplace Transform s-step and Laplace Transform of the function *sh* (ξt)

If $s < 0$

$$\mathcal{L}_s^{\star}(f_\xi)(\omega) = det(M^*)\mathcal{L}(sh(\xi t))(\omega)\left(e^{\omega s} + e^{2\omega s} + e^{3\omega s} + e^{4\omega s} + e^{5\omega s}\right).$$

If $s \geq 0$

$$\begin{aligned} \mathcal{L}_s^{\star}(f_\xi)(\omega) &= det(M^*)\mathcal{L}(sh(\xi t))\left[ch(s\xi) + ch(2s\xi) + ch(3s\xi) + ch(4s\xi) + ch(5s\xi)\right] \\ &+ det(M^*)\mathcal{L}(ch(\xi t))\left[sh(s\xi) + sh(2s\xi) + sh(3s\xi) + sh(4s\xi) + sh(5s\xi)\right]. \end{aligned}$$

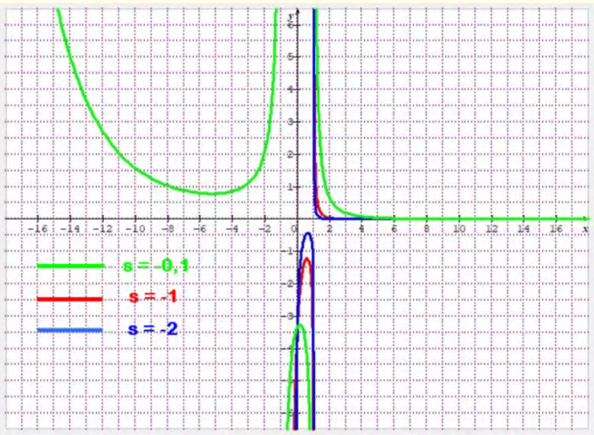

Figure 8: The graph of the Star-Laplace Transform *s*-step $L^{\star}_s f_1(t)$, if $s < 0$.

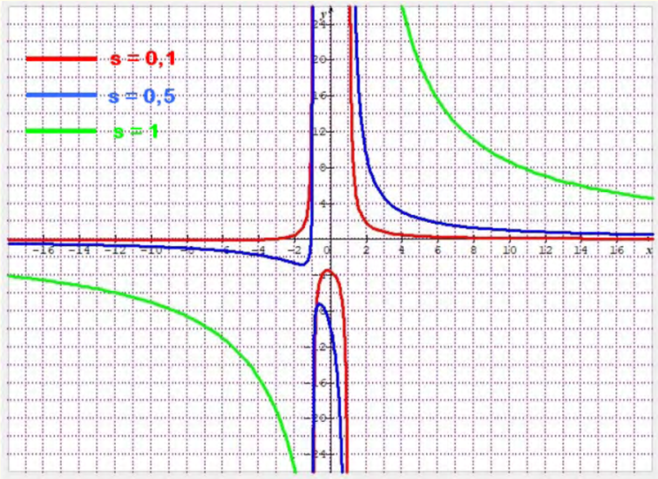

Figure 9: The graph of the Star-Laplace Transform s-step $L^*_s f_1(t)$, if $s > 0$.

References

1. Chaffar Mohamed Moktar. "Star with α coefficient in the set of real numbers". Journal of Applied and Computational Mathematics 9 (2020).
2. Chaffar, Mohamed Moktar. "Matrix representation of a star". Journal of Applied and Computational Mathematics 9 (2020).
3. Chaffar Mohamed Moktar and Hassen Ben Mohamed. "Star-jacobian Matrix of Star with α Coefficient". International Journal of Mathematics Trends and Technology 67.2 (2021): 85-102.
4. Ivan Kyrchel. "The Theory of the Column and Row Determinants in a Quaternion Linear Algebra". Advances in mathematics Research 15 (2012).
5. Robert Vein and Paul Dale. "Determinants and Their Applications in Mathematical Physics". Mathematical Sciences Springer 134 (2011).
6. Ron Larson and Bruce H Edwards. Calculus, Brooks/Col, Cengage Learning (2009).
7. Aneshkumar M. "Students' Understanding of Solving a System of Linear Equations Using Matrix Methods: A Case Study". Int J Edu Sci 21.1-3 (2018): 124-134.
8. BO Berndtsson. "Fourier and Laplace transforms".