PriMera Scientific Engineering
Volume 1 Issue 1 September 2022
https://doi.org/10.56831/PSEN-01-002

Review Article

Developing Model of Bridge to Identification Eigen-Frequencies of Bridges Using the Theory of Continuous Systems

Triantafyllos Makarios[1]* and Vassilios Lekidis[2]

[1]*Department of Civil Engineering, Aristotle University of Thessaloniki, Greece*

[2]*Hellenic Institute of Engineering Seismology and Earthquake Engineering, O.A.S.P., Greece*

***Corresponding Author:** Triantafyllos Makarios, Professor, Department of Civil Engineering, Aristotle University of Thessaloniki, Greece.

Citation: Triantafyllos Makarios., et al. "Developing Model of Bridge to Identification Eigen-Frequencies of Bridges Using the Theory of Continuous Systems". PriMera Scientific Engineering 1.1 (2022): 04-11.

Received: August 11, 2022

Published: August 29, 2022

Abstract

In the present article, an ideal equivalent three Degrees of Freedom (DoF) system of a one-bay-bridge (that is supported on elastometallic bearings) that has distributed stiffness and mass along its length is given. From the naturally point of view, the bridge has infinity number of degrees of freedoms, but based on the free vibration study, using partial differential equation, a mathematical ideal Three-Degrees of Freedoms system is obtained, where its ideal mass matrix is analytically written at specific mass locations on the bridge. Thus, using this abovementioned three Degree of Freedom system, the first three fundamental mode-shapes of the real bridge are identifying. Moreover, we consider the 3x3 mass matrix, we can attempt an estimation of the future feasible damages on the bridge, if a known technique for the identification of dynamic characteristics applied. Furthermore, the way of installation of a local network of three uniaxial accelerometers must be compatible with the abovementioned Three-Degrees of Freedom. It is noting worthy, this technique can be applied on bridges, where the sense of concentrated mass is fully absent.

Abbreviations

DoF: Degree of Freedom.

Introduction

The design of special structures such as large bridges, pipes etc., demands the use of reliable input data for the analysis, effective methods and realistic models of analysis. The modeling of the true behavior of a civil engineering structure is the first and the most difficult step in dynamic or seismic analysis. Among the important parameters to be defined are the mass distribution, the damping characteristics, the stiffness of the main load resisting system, the influence of secondary elements and interaction phenomena. The instrumentation of these structures contributes toward a better understanding of their dynamic performance, as well as a more accurate and reliable prediction of the earthquake resistance of such large-scale structures. In experimental analysis, the classical ways to estimate the modal parameters of the whole structural system, as well as of its structural elements, is either to excite the structure artificially (using, e.g., vibrators, heavy vehicles) or to evaluate the recordings obtained from a weak or strong ground motion.

A considerable amount of research has been carried out in the field of experimental and analytical study of various structures -and especially bridges- in the last decade (Ewins & Griffin, 1981; Gates & Smith, 1984; Haibach, 1986; Imai et al, 1989; Karabinis, 1991; Hong & Yun, 1993; Johanson, 1993; Ewins, 1995; Lekidis et al, 2005).

When a structure is excited dynamically, the response of the structure can be recorded, and from the response, the parameters describing the dynamic properties of the structure, the modal parameters, might be identified. This process, usually called experimental modal analysis, may be carried out in different ways depending on the size of the structure, the type of loading and the configuration of the instrumentation system.

Investigations to assess the dynamic characteristics of bridges using analytical and experimental methods contribute also to maintain bridges over passing big national roads. By comparing the results of both approaches and adjusting the analytical model, a more realistic modelling of the bridge can be achieved. As a result, a more accurate and reliable prediction of the earthquake resistance of the bridge is possible.

In this research a mathematic ideal three degrees of Freedom system that is equivalent to the modal behaviour of an infinity number of degrees of freedom of a bridge is given here. This ideal three DoF system can be used in installation of a local network of three uniaxial accelerometers on one bay bridge, that the sense of the concentrated masses is absent. This point is a basic obstacle that often appears during the instrumentation of such bridges or steel stairs (Makarios, 2020a; Manolis et al, 2015; Makarios et al, 2015 and 2017) or wind energy power (Makarios and Baniotopoulos, 2014 and 2015) to identify the vibration mode shapes using records of response accelerations) due to ambient excitation (Makarios, 2012 and 2013).

Eigen-Modal Analysis of bridge without damping

Using the Theory of Continuous Systems (Clough and Penzien, 1975; Copra, 1995; Makarios, 2020b), consider a straight beam that is loaded by an external continuous dynamic loading $p_z(x,t)$, with reference to a Cartesian three-dimensional reference system *oxyz*, (Figure 1). The bridge has a distributed mass *m(x)* per unit length, which in the special case of uniform distribution is taking as (in tons per meter tn/m). Moreover, according to Bernoulli Technical Bending Theory, the bridge has section flexural stiffness $EI_y(x)$, where in the specific case of a uniform distribution of the stiffness it is taking as $EI_y(x)=EI_y$, where *E* is the material modulus of elasticity and I_y is the section moment of inertia around y-axis (Figure 1). Furthermore, we are examining a such bridge that has constant value of distributed mass along its length and, also, has constant value of distributed section flexural stiffness EI_y. Due to the fact that the bridge mass is continuously distributed, this bridge has infinity number of degrees of freedom for vibration along the vertical *oz*-axis. To mathematically write the motion equation of this bridge, consider an infinitesimal part, at location *x* from the origin o, that has been isolated by two very nearest parallel sections. The infinitesimal length of this part is namely as the *dx*. On this infinitesimal length, we notice the flexural moment *M(x,t)*, the shear force *Q(x,t)* with their differential increments, while the axial force N(*x,t*) is ignored, because it doesn't affect the vertical beam vibration along z-axis. Moreover, noted the resulting force $P_z(x,t)$ of the external dynamic loading. Therefore, we can write:

$$P_z(x,t)=p_z(x,t)\cdot dx \qquad (1)$$

Where the resulting force $P_z(x,t)$ acts at the total beam infinitesimal part.

Moreover, according to D'Alembert Principle, the resulting inertia force $F_a(x,t)$ is resulted, where:

$$F_a(x,t)=(-\bar{m}\cdot dx)\cdot \partial^2 u_z(x,t)/\partial t^2 \Rightarrow F_a(x,t)=(-\bar{m}\cdot dx)\cdot \ddot{u}_z(x,t) \qquad (2)$$

Here we agree that the time derivatives of the displacements are going to symbolize with full stops, while the spatial derivatives of the displacements are going to symbolize with accent. Furthermore, the damping and the second order differential of the bridge are ignored, so the force equilibrium on the infinitesimal part of the beam along z-axis is giving:

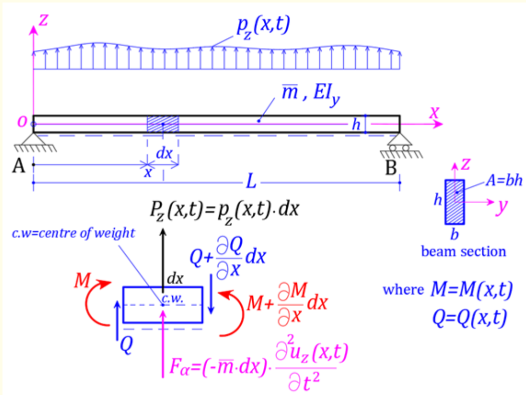

Figure 1: One bay bridge with distributed section flexural stiffness and mass.

$$\sum F_z = 0 \quad \Rightarrow \quad Q + P_z(x,t) - \left(Q + \frac{\partial Q}{\partial x}dx\right) + F_a(x,t) = 0 \quad \Rightarrow \quad \frac{\partial Q}{\partial x} = p_z(x,t) - \bar{m}\cdot\ddot{u}_z(x,t) \qquad (3)$$

Next, the moment equilibrium with reference to centre of weight (c.w.) of the infinitesimal part of the beam (see Figure 1) is giving:

$$\sum M_y = 0 \quad \Rightarrow \quad M + Q\cdot\frac{dx}{2} + \left(Q + \frac{\partial Q}{\partial x}dx\right)\cdot\frac{dx}{2} - \left(M + \frac{\partial M}{\partial x}dx\right) = 0 \Rightarrow \quad Q = \frac{\partial M}{\partial x} \qquad (4)$$

According to Euler-Bernoulli Technical Bending Theory (where the shear deformations are ignored) it is well-known that the following basic equation is written:

$$M(x,t) = EI_y\cdot\frac{\partial^2 u_z(x,t)}{\partial x^2} \qquad (5)$$

Equations (4) and (5) are inserting into equation (3), thus the motion equation for the examined bridge is written:

$$\frac{\partial^2 M}{\partial x^2} = p_z(x,t) - \bar{m}\cdot\ddot{u}_z(x,t) \Rightarrow \qquad \frac{\partial^2}{\partial x^2}\left(EI_y\cdot\frac{\partial^2 u_z(x,t)}{\partial x^2}\right) = p_z(x,t) - \bar{m}\cdot\ddot{u}_z(x,t) \qquad \Rightarrow$$

$$\bar{m}\frac{\partial^2 u_z(x,t)}{\partial t^2} + EI_y\frac{\partial^4 u_z(x,t)}{\partial x^4} = p_z(x,t) \qquad \Rightarrow \qquad \bar{m}\cdot\ddot{u}_z(x,t) + EI_y\cdot u_z''''(x,t) = p_z(x,t) \qquad (6)$$

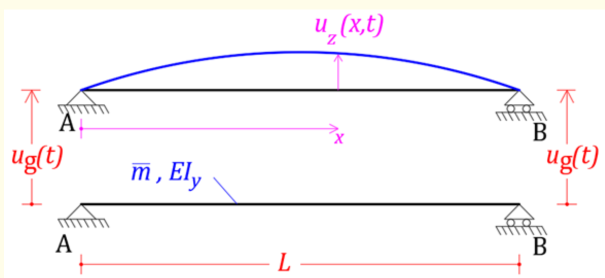

Figure 2: One bay bridge subjected with the same vertical ground motion u_g *(t)* on the two end-supports.

Equation (6) is a partial differential equation that describes the motion $u_z(x,t)$ of the bridge that is loaded with the external dynamic loading $p_z(x,t)$. To arise a unique solution from Eq.(6), the support conditions must be used at the two bridge ends. It is worth noting that the classical case of a bridge with distributed mass and section flexural stiffness, under external vertical excitation (Figure 2) on the two supports is mathematically equivalent with the vibration that is described by equation (6). Indeed, in the case of the Figure (2), the total displacement $u_z^{tot}(x,t)$ of the beam at x-location is written:

$$u_z^{tot}(x,t) = u_g(t) + u_z(x,t) \qquad (7)$$

Where $u_g(t)$ is the displacement at the base that is the same for the two end-supports. But, it is known that the inertia forces of the bridge are depended by the total displacement $u_z^{tot}(x,t)$, while the distributed dynamic loading is null, $p_z(x,t) = 0$. So, the equation (3) is transformed into:

$$\frac{\partial Q}{\partial x} = p_z(x,t) - \bar{m} \cdot \frac{\partial^2 u_z^{tot}(x,t)}{\partial t^2} \quad \Rightarrow \quad \frac{\partial Q}{\partial x} = 0 - \bar{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} - \bar{m} \cdot \frac{\partial^2 u_z(x,t)}{\partial t^2} \qquad (8)$$

Following, equations (4-5) are inserting into equation (8), thus we are given:

$$\frac{\partial^2 M}{\partial x^2} = -\bar{m} \cdot \left(\frac{\partial^2 u_g(t)}{\partial t^2} + \frac{\partial^2 u_z(x,t)}{\partial t^2}\right) \quad \Rightarrow \quad \bar{m}\frac{\partial^2 u_z(x,t)}{\partial t^2} + EI_y \frac{\partial^4 u_z(x,t)}{\partial x^4} = -\bar{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} \qquad (9)$$

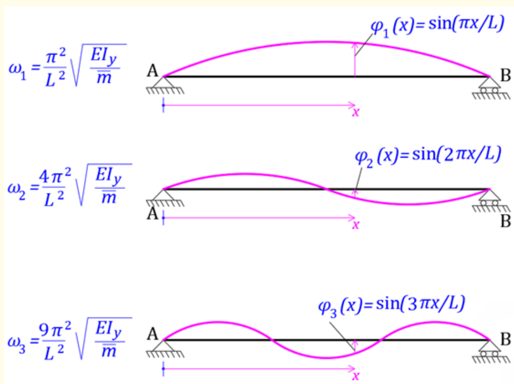

Figure 3: Modal analysis of Continuous bridge. The three first mode-shapes.

If compare the two equations (6) and (9), we notice that the bridge vibration due to vertical motion of the two supports is mathematically equivalent to the undamped vibration of the same bridge where the two supports are fixed and the bridge is loaded with the equivalent distributed dynamic loading $p_{eq}(x,t)$:

$$p_{eq}(x,t) = -\bar{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} \qquad (10)$$

In the case of the bridge free vibration, we consider the first part of equation (9) that must be null:

$$\bar{m}\frac{\partial^2 u_z(x,t)}{\partial t^2} + EI_y \frac{\partial^4 u_z(x,t)}{\partial x^4} = 0 \qquad (11)$$

As proved in previous papers (Makarios, 2020a and 2002b) the solution of Eq.(11) arrive ateigen-frequency ω_n is directly arise for each n-value.

$$\omega_n = \frac{n^2 \cdot \pi^2}{L^2} \cdot \sqrt{\frac{EI_y}{\bar{m}}} n = 1,2,3,\ldots \qquad (12)$$

Therefore, the vibration mode-shape of the examined bridge arises:

$$\varphi_n(x) = C_1 \sin\beta x = C_1 \sin\frac{n\cdot\pi\cdot x}{L} n = 1,2,3,\dots \quad (13)$$

The value of C_1 is arbitrary, and we usually get it equal to unit. Thus, for each value of parameter *n*, a mode-shape with its eigen-frequency is resulted. The fundamental (first) mode-shape results for *n = 1*, which shows a half sinusoidal wave, the second mode-shape shows a three sinusoidal wave, etc. (Figure 3). The order of the eigen-frequencies is ω_1, $\omega_2 = 4\omega_1$, $\omega_3 = 9\omega_1$.

The Ideal Equivalent Three Degrees of Freedom System

At bridges where the fundamental mode shape does not activate above than 90% of the total bridge mass, we ask to consider the three first mode shapes. Thus, for this purpose, we must define an ideal equivalent three degrees of freedom beam, which is going to give the three mode-shapes of the examined beam. Therefore, which is the ideal three degrees of freedom system, where its three mode-shapes coincide with the real first three mode-shapes of the bridge with distributed mass and flexural stiffness?

To answer the above-mentioned question, consider a weightless bridge with length *L* and constant section along its length, where carry three concentrated masses that each one has the same mass-value m_{eq}, located per distance 0.25*L*, between one to one, and each one mass has a vertical degree of freedom (Figure 4).

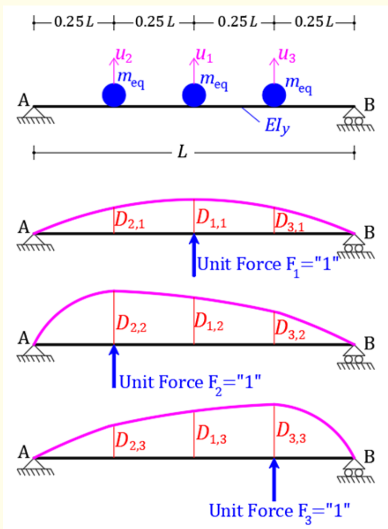

Figure 4: The equivalent Three-Degree of Freedom system.

The bridge displacement vector u of the three degrees of freedom, as well as the diagonal bridge mass matrix *m* are written:

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \qquad \mathbf{m} = \begin{bmatrix} m_{eq} & 0 & 0 \\ 0 & m_{eq} & 0 \\ 0 & 0 & m_{eq} \end{bmatrix} \quad (14)$$

Moreover, the bridge flexibility matrix *f* can be calculated using a suitable method (see in Figure 4), and the inverse matrix gives the stiffness matrix *k* of the three degrees of freedom bridge.

$$= \begin{bmatrix} D_{1,1} & D_{1,2} & D_{1,3} \\ D_{2,1} & D_{2,2} & D_{2,3} \\ D_{3,1} & D_{3,2} & D_{3,3} \end{bmatrix} = \frac{L^3}{48EI_y} \cdot \begin{bmatrix} 1 & 0.6875 & 0.6875 \\ 0.6875 & 0.5625 & 0.4375 \\ 0.6875 & 0.4375 & 0.5625 \end{bmatrix} \quad (15)$$

$$\boldsymbol{k} = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} \\ k_{2,1} & k_{2,2} & k_{2,3} \\ k_{3,1} & k_{3,2} & k_{3,3} \end{bmatrix} = \frac{48EI_y}{L^3} \cdot \begin{bmatrix} 18.285714 & -12.571429 & -12.571429 \\ -12.571429 & 13.142857 & 5.142857 \\ -12.571429 & 5.142857 & 13.142857 \end{bmatrix} \qquad (16)$$

The equations of motion for the case of the free undamped vibration of the ideal bridge is given in matrix form:

$$\mathbf{m}\,\ddot{\mathbf{u}}(t) + \mathbf{k}\,\mathbf{u}(t) = \mathbf{0} \qquad (17)$$

The eigen-problem is written:

$$(\mathbf{k} - \boldsymbol{\omega}_n^2\mathbf{m})\boldsymbol{\varphi}_n = \mathbf{0} \qquad n = 1,2,3. \qquad (18)$$

Where, the eigen-frequencies $\boldsymbol{\omega}_n$ and the three mode-shapes $\boldsymbol{\varphi}_n$ are known by equation (13) and Figure 4. Thus, the unique unknown parameter is the mass m_{eq}. Therefore,

$$\det(\mathbf{k} - \boldsymbol{\omega}_1^2\mathbf{m}) = 0 \qquad \Rightarrow \qquad (19)$$

$$m_{\mathrm{eq}}^3 + A \cdot m_{\mathrm{eq}}^2 + B \cdot m_{\mathrm{eq}} + C = 0 \qquad (20)$$

Where,

$$A = -\frac{k_{11} + k_{22} + k_{33}}{\omega_1^2}, \quad B = \frac{k_{11}k_{33} + k_{11}k_{22} + k_{22}k_{33} - k_{12}^2 - k_{13}^2 - k_{23}^2}{\omega_1^4}$$

$$C = -\frac{k_{11}k_{22}k_{33} + 2k_{12}k_{13}k_{23} - k_{11}k_{23}^2 - k_{22}k_{13}^2 - k_{33}k_{12}^2}{\omega_1^6}$$

The numerical solution of equation (20) gives three roots for parameter m_{eq}, where only the first root is acceptable, because the other two values rejected since do not have natural meaning (appear values greater from the total beam mass $\bar{m}L$). Thus, the only one acceptable root is given:

$$m_{eq} = 0.25 \cdot (\bar{m}L) \qquad (21)$$

Therefore, inserting the ideal equivalent mass m_{eq} by equation (21) at three degrees of freedom system of Figure 4, the three eigen-frequencies and mode-shapes coincide with the real values of the initial beam that has distributed mass and flexural stiffness.

It is noting that in the case where we would like the beam-bridge to be divided at ten same beam-items with item length L_i, then the equivalent mass of each item has to the real value, namely:

$$m_{eq} = \bar{m}\,L/10 = \bar{m}L_i \qquad (22)$$

If we consider other bridge models that we can see in Figure 5 and working similar according to all above mentioned mathematic analysis, the equivalent mass for each case of the bridge models is given into Figure 5.

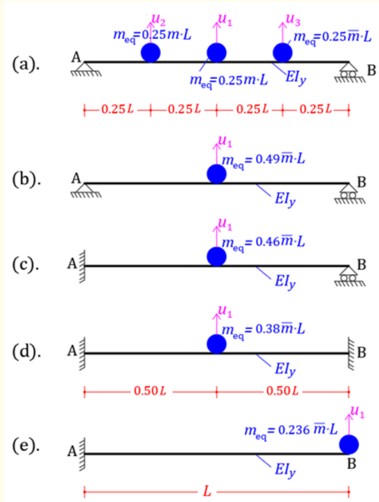

Figure 5: The equivalent mass for various model bridge cases.

Conclusions

The present article has presented a mathematic ideal three degrees of freedom system that is equivalent to the modal behavior of the one bay bridge with distributed mass and flexural stiffness along its length. This ideal three degrees of freedom system can be used in instrumentation of a such bridge, which does not possess concentrated masses. In the framework of the identification of mode-shapes of one bay bridge, the equivalent mass by equation (24) permits to locate accelerometers per 0.25*L* (as it shown at Figure 4) and there measure the response acceleration histories, in order to calculate the real first three mode shapes of the bridge. Last but not least, in Figure 5 are given the values of the equivalent mass for other bridge models, such as the vibration of these models to be equal with the real, naturally vibration bridge that has distributed mass and flexural stiffness along its length. This method contributes to instrumentation improvement for bridges, so as to get the best recordings, from permanent mobile networks. The proposed methodology provides the maintenance departments of structural companies a useful tool for the continuous assessment of the structural condition of the bridges under their surveillance.

References

1. Chopra A. “Dynamics of Structures. Theory and Applications to Earthquake Engineering”, International edition, Prentice-Hall, Inc., Englewood Cliffs. New Jersey (1995): 07632.
2. Clough R and Penzien J. “Dynamics of Structures”. Mc Graw-Hill (1975).
3. Ewins DJ. “Modal Testing: Theory and Practice”. Research Studies Press Ltd, England (1995).
4. Ewins DJ and Griffin J. “A State-of-the-art Assessment of Mobility Measurement Techniques- Results for the Mid-Range Structures”. Journal of Sound and Vibration 78.2 (1981): 197-222.
5. Gates J and Smith M. “Results of Ambient Vibration Testing of Bridges”. Proceedings of the 8th Word Conference on Earthquake Engineering, IV, San Francisco (1984): 873-880.
6. Haibach E. “Measurement and Interpretation of dynamic loads on bridges”. Technical Steel Research, Synthesis Report to C.E.C 9759 (1986): 1-98.
7. Hong KS and Yun CB. “Improved Method for Frequency Domain Identifications of Structures”. Journal of Engineering Structures 15.3 (1993): 179-188.
8. Imai H., et al. “Fundamentals of System Identification in Structural Dynamics”. Technical Report NCEER -89-0008 (1989): 1.72.
9. Johanson R. “System Modeling and Identification”. Prentice-Hall (1993): 1-526.

10. Karabinis AI. "Dynamic characteristics of a typical Prestressed Highway Bridge". International Conference on Dimensions in Bridges and Flyovers, Singapore (1991): 143-149.
11. Lekidis V., et al. "Evaluation of dynamic response and local soil effects of the Evripos cable-stayed bridge using multi-sensor monitoring systems". Engineering Geology 79.1-2 (2005): 43-59.
12. Makarios T. "Identification of the mode shapes of spatial tall multi-storey buildings due to earthquakes. The new "modal time-histories" method". Journal of the Structural Design of Tall & Special Buildings 21.9 (2012): 621-641.
13. Makarios T. "Identification of building dynamic characteristics by using the modal response acceleration time-histories in the seismic excitation and the wind dynamic loading cases". CHAPTER 4 of Book "Accelerometers; Principles, Structure and Applications". Nova Science Publisher (2013).
14. Makarios T and Baniotopoulos C. "Wind Energy Structures: Modal Analysis by the Continuous Model Approach". Journal of Vibration and Control 20.3 (2014): 395-405.
15. Makarios T and Baniotopoulos C. "Modal analysis of wind turbine tower via its continuous model with partially fixed foundation". International Journal of Innovative Research in Advanced Engineering 2.1 (2015): 14-25.
16. Makarios TK., et al. "Modelling and identification of the dynamic response of an existing three-story steel stairway". COMPDYN, 5th ECCOMAS Thematic Conference on Computational Methods in Structural Dynamics and Earthquake Engineering, Crete Island, Greece (2015): 25-27.
17. Makarios T., et al. "Identification of dynamic characteristics of a continuous system: case study for a flexible steel stairway". 16th World Conference on Earthquake, WCEE 16 (2017): 9-13.
18. Makarios T. "Identification of eigen-frequencies and Mode-shapes of beams with Continuous distribution of mass and elasticity and for various Conditions at supports". Chapter of Book entitled "Number Theory and its Applications". INTECHOPEN (2020a).
19. Makarios T. "Identification of Mode-Shapes and Eigen-Frequencies of bi-hinge Beam with Distributed Mass and Stiffness". Journal of Civil Engineering and Construction 9.3 (2020b): 119-126.
20. Manolis GD., et al. "Mode shape identification of an existing three-story flexible steel stairway as a continuous dynamic system". Journal Theoretical and Applied Mechanics 42.3 (2015): 151-16.